

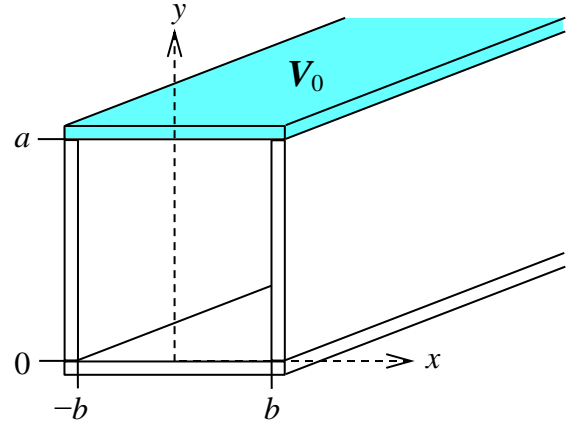
David J. Griffiths, *Introduction to Electrodynamics*, 3rd ed.

Problem 3.47

For the infinite rectangular pipe in Example 3.4, suppose the potential on the bottom ($y = 0$) and the two sides ($x = \pm b$) is zero, but the potential on the top ($y = a$) is a nonzero constant V_0 . Find the potential inside the pipe.

[Note: This is a rotated version of problem 3.14(b), but set it up as in Example 3.4 using sinusoidal functions in y and hyperbolics in x . It is an unusual case in which $k = 0$ must be included. Begin by finding the general solution to Equation 3.26 when $k = 0$. For further discussion see

S. Hassani, *Am. J. Phys.* **59**, 470 (1991).]



Ok, first assert that the potential is separable, $V = X(x)Y(y)$, and separate the variables in the usual way.

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0$$

The next step is to recognize that the placement of the coordinate axis in the middle of the pipe allows both sines and cosines to contribute to the solution for $V(x, y)$ along the x direction. It also means that the $k = 0$ solution can contribute to V . To understand this, note that we couldn't accept the $k = 0$ solution in Example 3.4 because only sines were allowed along the y direction, and the sine of zero is zero, so that term in the summation could only add a zero and we disregarded it. To make that statement even clearer, let's look again at Example 3.4. If $k = 0$ contributes to that solution, then we must include linear terms like Ex and Fy :

$$\begin{aligned} \frac{1}{X} \frac{d^2 X}{dx^2} &= +k^2 \Rightarrow X = Ae^{kx} + Be^{-kx} + Ex \\ \frac{1}{Y} \frac{d^2 Y}{dy^2} &= -k^2 \Rightarrow Y = C \sin ky + D \cos ky + Fy \end{aligned}$$

For Example 3.4, the potential went from V_0 at $x = -b$ to V_0 at $x = +b$, so any solution for X must be symmetric about the origin, and the linear term Ex is antisymmetric, thus $E = 0$. In that example we later learn that $A = B$ and the hyperbolic cosine (the symmetric function) is the one that satisfies the boundary conditions and contributes to the potential V . Furthermore, the function Y must start at zero (at $y = 0$) and return to zero at $y = a$, something the linear term Fy cannot do, thus $F = 0$. That's why Example 3.4 never mentions the linear terms.

Now in this problem, problem 3.47, I'm going to ignore the hint and go with my gut feeling that the solution should be sinusoidal functions in x and hyperbolics in y :

$$\begin{aligned} \frac{1}{X} \frac{d^2 X}{dx^2} &= -k^2 \Rightarrow X = A \sin kx + B \cos kx + Ex \\ \frac{1}{Y} \frac{d^2 Y}{dy^2} &= +k^2 \Rightarrow Y = Ce^{ky} + De^{-ky} + Fy \end{aligned}$$

We can see that the function X must start at zero (at $x = -b$) and return to zero at $x = b$, something the linear term Ex cannot do, thus $E = 0$. The $k = 0$ part of X is thus simply a constant:

$$X_{k=0} = A \sin(0) + B \cos(0) + 0 = B$$

However, a linear term *could* contribute along the y direction. This $k = 0$ portion of the solution would look like:

$$Y_{k=0} = Ce^0 + De^{-0} + Fy = Fy + G$$

where $G = C + D$. Now the $k = 0$ portion of the *entire* potential V must therefore satisfy these boundary conditions:

$$V_{k=0} = X_{k=0} Y_{k=0} = B(Fy + G) \text{ in general, and then at the boundaries:}$$

$$V_{k=0}(x, 0) = 0 = B(0 + G) \Rightarrow G = 0$$

$$V_{k=0}(x, a) = V_0 = BFa \Rightarrow BF = \frac{V_0}{a}, \text{ so that}$$

$$V_{k=0}(x, y) = \frac{V_0}{a} y$$

The complete potential, with every possible value of k , must be a sum of the $k = 0$ term and a nonzero k term:

$$V(x, y) = V_{k=0} + V_{k>0} = \frac{V_0}{a} y + V_{k>0}$$

To find the $V_{k>0}$ part, subtract the $V_{k=0} = V_0 y/a$ part from V . As you can see from the diagram at right, this leaves a situation very much like Example 3.4, except instead of constant potentials at $x = \pm b$, there are potentials that are linear in y . Note this means that for the $k > 0$ portion of the potential, we should choose sinusoidal solutions in the y direction and hyperbolic in the x direction. Setting $Y_{k>0}$ equal to zero at $y = 0$ and $y = a$, we find:

$$Y_{k>0} = A_n \sin\left(\frac{n\pi}{a} y\right) \text{ where } n = 1, 2, 3, \dots$$

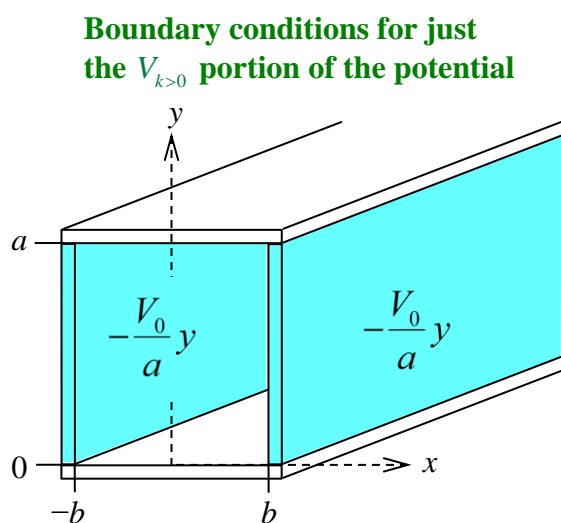
Noting that $X_{k>0}$ must be symmetric, and its constants must be the same as those in $Y_{k>0}$ so that they'll cancel out when you take the second derivatives, we find

$$X_{k>0} = C \left(e^{\frac{n\pi}{a} x} + e^{-\frac{n\pi}{a} x} \right) = 2C \cosh\left(\frac{n\pi}{a} x\right)$$

Combining these two functions, the $k > 0$ portion of the potential looks like:

$$V_{k>0} = X_{k>0} Y_{k>0} = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{a} y\right)$$

We can find these constants C_n by applying the boundary condition at $x = \pm b$ and using Fourier's trick:



Set the potential equal to $-V_0 y/a$ at the boundaries:

$$V_{k>0}(b, y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi}{a} b\right) \sin\left(\frac{n\pi}{a} y\right) \stackrel{\text{set}}{=} -\frac{V_0}{a} y$$

Apply Fourier's trick to get rid of the infinite sum:

$$\int_0^a \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi}{a} b\right) \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right) dy = \int_0^a \left(-\frac{V_0}{a} y\right) \sin\left(\frac{m\pi}{a} y\right) dy$$

Evaluate the integrals:

$$\sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi}{a} b\right) \begin{cases} 0 & \text{if } n \neq m \\ \frac{a}{2} & \text{if } n = m \end{cases} = -\frac{V_0}{a} \left[\left(\frac{a}{m\pi}\right)^2 \sin\left(\frac{m\pi}{a} y\right) - \frac{a y}{m\pi} \cos\left(\frac{m\pi}{a} y\right) \right]_0^a$$

Simplify the expressions for the case where $n = m$:

$$C_m \cosh\left(\frac{m\pi}{a} b\right) \left(\frac{a}{2}\right) = -\frac{V_0}{a} \left[\left[0 - \frac{a^2}{m\pi} \cos\left(\frac{m\pi}{a} y\right) \right] - [0 - 0] \right]$$

Note that $\cos(m\pi) = (-1)^m$:

$$C_m \cosh\left(\frac{m\pi}{a} b\right) \left(\frac{a}{2}\right) = \frac{V_0 a}{m\pi} (-1)^m$$

Now solve for C_m :

$$C_m = \frac{2V_0}{m\pi} \frac{(-1)^m}{\cosh\left(\frac{m\pi b}{a}\right)}$$

We can now add the two potentials $V_{k=0}$ and $V_{k>0}$ to get our final answer:

$$V(x, y) = \frac{V_0}{a} y + \frac{2V_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\cosh\left(\frac{n\pi}{a} x\right)}{\cosh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi}{a} y\right)$$

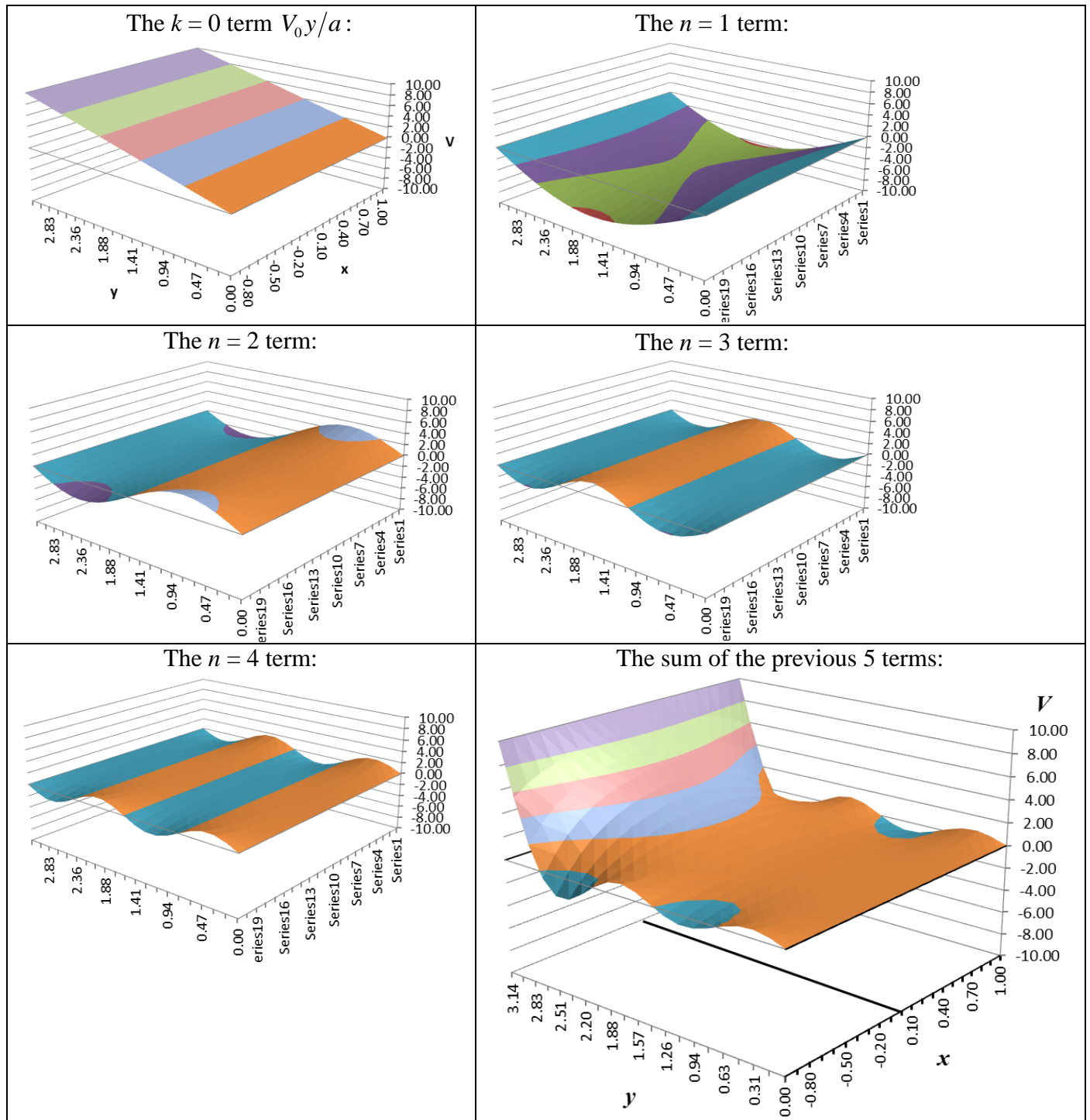
Note that in this case both odd and even values of n contribute to the potential. I've graphed the zeroth and first 4 values of n on the next page.

I haven't worked this out, but Griffiths says you can get an equivalent expression by using sinusoidal functions of x and hyperbolic functions of y , presumably avoiding the $k = 0$ term:

$$V(x, y) = -\frac{2V_0}{b} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\sinh(\alpha_n y)}{\sinh(\alpha_n a)} \cos(\alpha_n x), \text{ where } \alpha_n = \frac{(2n-1)\pi}{2b}$$

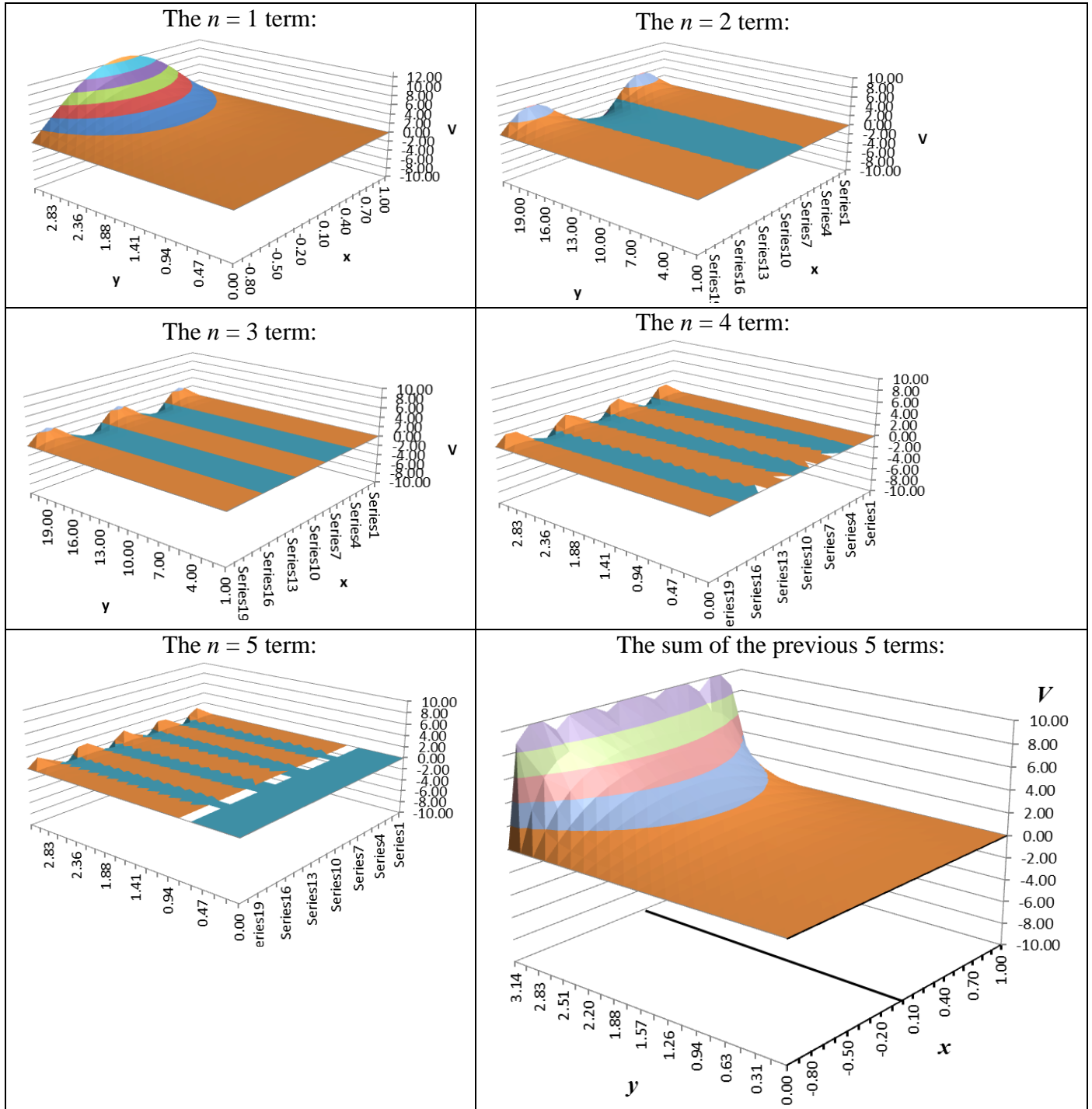
Note that I added a minus sign in front of the equation when compared with the equation displayed on p.159 of the text. I discovered the minus sign was needed in order to get the spreadsheet summation of the first five terms to produce the correct results. You can see the spreadsheet results on the last page of this document. I haven't worked out the analytical solution to discover why the minus sign is necessary, or whether Griffiths was mistaken when he omitted it.

The first few terms of $V(x, y) = \frac{V_0}{a} y + \frac{2V_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\cosh\left(\frac{n\pi}{a} x\right)}{\cosh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi}{a} y\right)$:



Note that this approach does a good job pinning the potential to V_0 at $y = a$, but not such a good job nailing it to zero at $x = \pm b$. The corners at $y = a$ are especially rough.

The first few terms of $V(x, y) = -\frac{2V_0}{b} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\sinh(\alpha_n y)}{\sinh(\alpha_n a)} \cos(\alpha_n x)$, where $\alpha_n = \frac{(2n-1)\pi}{2b}$:



Note that this approach does a good job pinning the potential to zero at $x = \pm b$, but not such a good job nailing it to V_0 at $y = a$. The corners at $y = a$ are especially rough.