A rod of length $L$ lies along the $x$ axis with its left end at the origin. It has a nonuniform charge density $\lambda = \alpha x$, where $\alpha$ is a positive constant. (a) What are the units of $\alpha$? (b) Calculate the electric potential at $A$.

(a) $\lambda = \frac{C}{m} = \alpha x = \frac{C}{m^2} \cdot m$ so $\alpha$ has units of [C/m$^2$]

(b) Assume $V = 0$ at $x = \infty$ and treat $V$ as the sum of point charge potentials from each bit of charge $dq$.

$V = \int K \frac{dq}{r} = \int_0^L K \frac{\lambda \, dx}{x + d} = \int_0^L K \frac{\alpha \, x \, dx}{x + d}$

look this up in math tables or rearrange:

$V = K \alpha \int_0^L \frac{(x + d - d) \, dx}{x + d} = K \alpha \int_0^L (1) \, dx - \frac{d \, dx}{x + d}$

$= K \alpha \left[ (L - 0) - d \left( \ln \frac{x + d}{10} \right) \right] = K \alpha \left[ L - d \left( \ln \frac{L + d}{d} \right) \right]$

$V = K \alpha \left[ L - d \ln \left( \frac{L + d}{d} \right) \right]$

Just for fun, let's compare this with a point charge at the midpoint of the rod.

$Q = \int_0^L \lambda \, dx = \alpha \int_0^L x \, dx = \frac{1}{2} \alpha L^2$ put this charge at the midpoint:

$V_{\text{pt charge}} = K \frac{Q}{r} = K \frac{\frac{1}{2} \alpha L^2}{d + \frac{1}{2} L}$

Now suppose $\alpha = 1.0 \times 10^{-9}$ C/m$^2$, $d = 0.30$ m, $L = 1.00$ m:

$V_{\text{exact}} = K \alpha \left[ L - d \ln \left( \frac{L + d}{d} \right) \right]$  

$= \left( 8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \right) \left( 1.0 \times 10^{-9} \text{ C/m}^2 \right) \left[ (1 \text{ m}) - (0.3 \text{ m}) \ln \left( \frac{1.3 \text{ m}}{0.3 \text{ m}} \right) \right]$  

$V_{\text{exact}} = 5.041 \text{ V}$

$V_{\text{pt charge}} = K \frac{\frac{1}{2} \alpha L^2}{d + \frac{1}{2} L} = \left( 8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \right) \frac{\frac{1}{2} \left( 1.0 \times 10^{-9} \text{ C/m}^2 \right) \left( 1.0 \text{ m} \right)^2}{0.30 + 0.50 \text{ m}} = 5.625 \text{ V}$

The point charge value is higher because in the exact case most of the charge is at the far end of the rod, but the point charge estimate moves it to the center of the rod.