Here, you’ll learn:

- what it means for a logic system to be finished
- some strategies for constructing proofs

Congratulations! Our system of propositional logic is now finished! It’s natural to wonder, however, what it means for a logical system to be finished. After all, we certainly haven’t proven all of the arguments there are to prove, and for all we know, there are more rules out there too. In what sense, then, is our system finished?

Finished Systems and System Choice

To understand what it means for a logical system to be finished, let’s note that our system of propositional logic is made up of the following components:

1) Connectors (e.g. “&,” “∨,” “→,” “↔,” and “~”)
2) Variables (e.g. “P,” “Q,” etc.)
3) Rules about how we can construct well-formed-formulas using the connectors and variables (e.g. “(□ & ◇) → △” is a well-formed-formula. “(□¬△ (&◇)” is not a well-formed-formula.)
4) A semantics that sets out the truth conditions for the well-formed-formulas (e.g. “□ & ◇” is true if and only if □ is true and ◇ is true.)
5) Inference rules that specify what well-formed-formulas can be inferred from other well-formed-formulas (e.g. “□ → △, □ ⊢ ◇)

We can say that a set of connectors is “expressively complete” if and only if it can be used to translate any sentence expressible within the language. Our system of propositional logic, with its set of connectors, variables, and rules for constructing formulas, is expressively complete.
A logical system is said to be “sound” if and only if only valid arguments expressible within the system can be proven within the system. In a sound logical system, the inference rules reliably track the semantics in such a way that if we can prove an argument then we know that whenever the premises of the argument are true the conclusion of the argument will be true as well. Provability is thus a guarantee of validity. You’ll be happy to know that our system of propositional logic is sound. A proof of its soundness is beyond our concern here, but you can take my word for it.

1) Connectors (e.g. “&,” “∨,” “→,” “↔,” and “¬”)
2) Variables (e.g. “P,” “Q,” etc.)
3) Rules about how we can construct well-formed-formulas using the connectors and variables (e.g. “(□ & ○) → △” is a well-formed-formula. “(□¬△ (□&○) is not a well-formed-formula.)
4) A semantics that sets out the truth conditions for the well-formed-formulas (e.g. “□ & ○” is true if and only if □ is true and ○ is true.)
5) Inference rules that specify what well-formed-formulas can be inferred from other well-formed-formulas (e.g. “ □ → ○, □ ⊢ ○”)

A logical system is said to be “complete” (not “expressively complete” but just “complete”) if and only if all valid arguments expressible within the system can be proven within the system. In a complete logical system, the inference rules reliably track semantics in such a way that if an argument is valid then it can be proven within the system. Validity is thus a guarantee of provability. You’ll be happy to know that our system of propositional logic is complete. A proof of its completeness is beyond our concern here, but once again you can take my word for it.

1) Connectors (e.g. “&,” “∨,” “→,” “↔,” and “¬”)
2) Variables (e.g. “P,” “Q,” etc.)
3) Rules about how we can construct well-formed-formulas using the connectors and variables (e.g. “(□ & ○) → △” is a well-formed-formula. “(□¬△ (□&○) is not a well-formed-formula.)
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5) Inference rules that specify what well-formed-formulas can be inferred from other well-formed-formulas (e.g. “ □ → ○, □ ⊢ ○”)

Because our system of propositional logic is both sound and complete, we know that only valid arguments expressible within the language can be proven within the system and we know that all valid arguments expressible within the language can be proven within the system.

1 If we ever have to choose between a system that’s sound but not complete and a system that’s complete but not sound, we should choose the sound but incomplete system every time. At the very least, we want our logical systems to be sound. The whole point of a set of inference rules, after all, is to establish the validity of arguments, and if the system is unsound then that goes out the window. An incomplete system, on the other hand, simply isn’t as useful as we’d like because its inability to prove some valid arguments would prevent us from concluding that an argument is invalid from the fact that it’s not provable within the system. As long as the system is sound, though, we can still conclude that an argument is valid from the fact that we can prove it, and so the system can still do some important work for us.
There’s really nothing more that we can ask of a logical system than that, and so our logic system is now finished! We don’t need to add any more rules to it in order for it to be able to prove everything we want it to prove.

In an important sense, our system of propositional logic could have been finished earlier because we could have omitted the derived rules altogether. After all, given the fact that the system containing both the primitive and the derived rules is complete (and it is) and given the fact that we’ve proven the derived rules using the primitive rules (and we did) then any argument that can be proven using the derived rules can be proven using the primitive rules alone.

For example, consider this argument:

\[(A \land B) \rightarrow (C \lor D), \neg C \land \neg D \vdash \neg A \lor \neg B\]

It may be proven using derived rules as follows:

1. \((A \land B) \rightarrow (C \lor D)\) P
2. \(\neg C \land \neg D\) P – want \(\neg A \lor \neg B\)
3. \(\neg (C \lor D)\) 2 DM
4. \(\neg (A \land B)\) 1, 3 MT
5. \(\neg A \lor \neg B\) 4 DM

And it may be proven using only primitive inference rules as follows:

1  1. \((A \land B) \rightarrow (C \lor D)\) P
2  2. \(\neg C \land \neg D\) P – want \(\neg A \lor \neg B\)
3  3. \(C \lor D\) PP – want contradiction
2  4. \(\neg C\) 2 &O
2,3  5. \(D\) 3, 4 &O
2  6. \(\neg D\) 2 &O
2,3  7. \(D \land \neg D\) 5, 6 &I
2  8. \(\neg (C \lor D)\) 3 – 7 &I
9  9. \(A \land B\) PP – want contradiction
1,9  10. \(C \lor D\) 1, 9 ➔O
1,2,9 11. \((C \lor D) \land \neg (C \lor D)\) 8, 10 &I
1,2 12. \(\neg (A \land B)\) 9-11 &I
13 13. \(\neg (\neg A \lor \neg B)\) PP – want contradiction
14 14. \(A\) PP – want contradiction
14 15. \(\neg A \lor \neg B\) 14 vI
13,14 16. \((\neg A \lor \neg B) \land \neg (\neg A \lor \neg B)\) 13, 15 &I
13 17. A 14 – 16 &O
18 18. \(\neg B\) PP – want contradiction
18 19. \(\neg A \lor \neg B\) 18 vI
13,18 20. \((\neg A \lor \neg B) \land \neg (\neg A \lor \neg B)\) 13, 19 &I
Let’s call the set of primitive inference rules “PR,” for “primitive rules” and let’s call our set of inference rules “PR + DR,” for “primitive rules plus derived rules.” Given that PR and PR + DR are both complete, we can’t decide between them on that basis. So which system is better?

Well, it depends upon what we mean by “better.” Pure logicians tend to prefer the smallest set of inference rules because there is theoretical interest in seeing how much can be done with little. Since every rule that’s contained in PR is contained in PR + DR, whereas PR + DR contains some rules that aren’t contained in PR, PR is a smaller system than PR + DR. Of course, this doesn’t mean that the proofs constructed in PR will be simpler than the proofs constructed in PR + DR. In fact, as we can see from the previous two proofs, the more rules we have, the more powerful each rule is, and the more powerful each rule is, the shorter and more intuitive the proofs using those rules can be. PR + DR allows for smaller proofs exactly because it isn’t the smaller system. PR forces us into great big proofs exactly because the inference set is so small.²

People interested in using logic in order to evaluate and construct arguments tend to prefer the most intuitive set of rules because those rules most neatly conform to our natural thought processes and so are able to reflect and guide those processes most effectively. I’ve assumed that we’re more interested in using logic than we are in studying the formal properties of logical systems for their own sake, and so we’ve developed a fairly intuitive set relatively robust rules, a set that is sufficiently large to capture most of our natural inferences but small enough to be manageable.³

² A parallel dynamic governs our set of connectors. Given the fact that \{\&, \lor, \rightarrow, \leftrightarrow, \neg\} is an expressively complete set of connectors for propositional logic, \{\neg, \lor\} is an expressively complete set of connectors as well. This is because formulas that use the connectors “\&,” “\rightarrow” and “\leftrightarrow” can be translated into equivalent formulas that use only the connectors “\lor” and “\neg.” Behold:

- \(P \& Q \equiv \neg(\neg P \lor \neg Q)\)
- \(P \rightarrow Q \equiv \neg P \lor Q\)
- \(P \leftrightarrow Q \equiv \neg(\neg P \lor Q) \lor (\neg Q \lor P)\)

Now, which set of connectors is simpler, \{\&, \lor, \rightarrow, \leftrightarrow, \neg\} or \{\neg, \lor\}? \{\neg, \lor\} is simpler because it does as much with less. The cost of this simplicity of resources, however, is shouldered by the complexity of the symbolizations.

³ Similarly, we’ve used a set of logical connectors that correspond (albeit roughly) to the connectors in common usage and that allow for relatively literal translations of sentences.
Proof Strategy

Now that we have a finished system of propositional logic in hand, we can practice using it. And to help us practice using it, there are some general proof strategies to bear in mind:

Strategy 1 – Hunt for the conclusion sitting somewhere.

We can apply this strategy if we’re lucky enough to see the conclusion sitting somewhere “gettable” in our premises or derived lines.

- If the conclusion is the consequent of an existing conditional, we might try to use →O.
- If the opposite of the conclusion is the antecedent of an existing conditional, we might try to use MT.
- If the conclusion is a disjunct of an existing disjunction, we might try to use ∨O.
- If the conclusion is a conjunct of an existing conjunction, we can deriving it using &O.

Strategy 2 – Note the main connector of the conclusion.

If we can’t see our conclusion sitting, “ready made,” in an existing line, we’ll need to build it ourselves. And we do this by bearing in mind the main connector of the conclusion.

- If main connector is “&,” we might use &I to derive the conclusion. This insight can start us trying to prove each of the conjuncts.
- If the main connector is “→,” we might use →I to derive the conclusion. This insight can have us take the antecedent of the conclusion as a provisional premise and attempt to prove the consequent of the conclusion from it.
- If the main connector is “↔,” we might use ↔I to derive the conclusion. This insight can start us trying to prove each of the component conditionals.
- If the main connector is “¬,” we might use ¬I to derive the conclusion. This insight can have us take the opposite of the conclusion as a provisional premise and attempt to prove a contradiction from it.
- If the main connector is “∨,” we probably won’t use ∨I to derive the conclusion, but we can take any of the following approaches:
  - We might try to prove the corresponding conditional and then use AR to convert the conditional to the disjunction (e.g. to prove P ∨ Q, we might prove ¬P → Q and use AR).
  - We might try to prove the corresponding negated conjunction and then use DM to convert the negated conjunction to the disjunction (e.g. to prove P ∨ Q, we might prove ¬(¬P & ¬Q) and use DM).
  - We might try to find or derive the elements that will allow us to prove the disjunction using CD (e.g. to prove P ∨ Q, we might find or derive lines of the form □ ∨ ○, □→P, ○→ Q and use CD).
Strategy 3 – Derive what we can.

If we can’t see the conclusion sitting anywhere “gettable,” and if we can’t see how to build the conclusion by focusing upon its main connector, we might want to draw some inferences from the information that we already have until we do suddenly see the conclusion or how to get it.

- We should do DM if we can. This will transform our negated conjunctions into disjunctions and our negated disjunctions into conjunctions.
- We should do &O and ↔O where we can, because this has the effect of simplifying lines and “releasing” information for future use.
- We should do ∨O were we can because this has the effect of “releasing” more information from the disjunction.
- We should do →O and MT if we can because both rules have the effect of “releasing” information from conditionals.
- We should do SD and CD if we can because this has the effect of “releasing” information from disjunctions and conditionals.
- We should do CH if we can because this consolidates information and gives us less to look at. Note: If we see conditionals with opposite antecedents or opposite consequents, we can use CN on one of them and then use CH.

Strategy 4 – Do a Reductio ad Absurdum.

If all else fails, we should consider using ~ I/O to derive the conclusion, taking the opposite of the conclusion and attempting to demonstrate that a contradiction follows from it.

Propositional Logic Practice

With our system of propositional logic firmly in hand, it’s time to practice! You can find my answers in the endnotes.

I) Equivalences

For each sentence, determine which of the sentences beneath it are equivalent to it. There may be more than one. (To determine if two sentences are equivalent to each other, symbolize the sentences and determine if the symbolizations are equivalent to teach other.)

1) “Being rich is not sufficient for being happy.”1

R = Some is rich.
H = Someone is happy.

a) “Someone can be rich but not happy.”
b) “Someone can be happy but not rich.”
c) “It isn’t the case that either someone isn’t rich or they are happy.”
d) “It isn’t the case that either someone is happy or they aren’t rich.”
e) “It isn’t the case that either someone isn’t happy or they are rich.”
f) “Not being rich is sufficient for not being happy.”

2) “Free will is not necessary for having moral responsibility.”

F = Someone has free will.
M = Someone has moral responsibility

a) “Someone can have free will without moral responsibility.”
b) “Someone can have moral responsibility without free will.”
c) “It isn’t the case that either someone doesn’t have free will or they do have moral responsibility.”
d) “It isn’t the case that either someone doesn’t moral responsibility or they do have free will.”
e) “Not having free will is necessary for not having moral responsibility.”
f) “Failing to have free will is not sufficient for failing to have moral responsibility.”

3) “Unless the class is full, she’ll take it.”

F = The class is full.
T = She’ll take the class.

a) “Either the class is full or she’ll take it.”
b) “If she doesn’t take the class then it’s full.”
c) “If she does take the class then it isn’t full.”
d) “It’s not the case that the class isn’t full and she won’t take it.”
e) “The class is full only if she doesn’t take it.”
f) “Her taking the class is not necessary for the class being full.”

4) “The town is neither friendly nor affordable.”

F = The town is friendly.
A = The town is affordable.

a) “The town’s being friendly is not necessary for it being affordable.”
b) “The town’s being friendly is not necessary for it failing to be affordable.”
c) “It’s not the case that either the town is friendly or it’s affordable.”
d) “Provided that the town is friendly, it isn’t affordable.”
e) “Provided that the town is affordable, it isn’t friendly.”
f) “It isn’t the case that unless the town is friendly, it’s affordable.”

II) English Arguments

Determine whether or not each of the following arguments is valid by symbolizing it and attempting to construct a valid proof.
1) “Morality is objective if either God exists or there is a moral law. Morality is objective only if people agree about what’s right and wrong. People don’t agree about what’s right and wrong. Therefore, neither God nor the moral law exist.”

O = Morality is objective.
G = God exists.
L = There is a moral law.
A = People agree about what’s right and wrong.

2) “Morality is objective only if either God exists or there is a moral law. Morality is objective only if people agree about what’s right and wrong. People don’t agree about what’s right and wrong. Therefore, neither God nor the moral law exist.”

O = Morality is objective.
G = God exists.
L = There is a moral law.
A = People agree about what’s right and wrong.

3) “Elliot is neither a chemistry major nor a physics major if he doesn’t have a lab. Elliot is either a chemistry major or a physics major. Provided that Elliot has a lab, he has to pay a special fee. Therefore, Elliot is assessed a special fee.”

C = Elliot is a chemistry major.
P = Elliot is a physics major.
L = Elliot has a lab.
F = Elliot is assessed a special fee.

4) “Elliot is neither a chemistry major nor a physics major if he doesn’t have a lab. Elliot is either a chemistry major or a physics major. Provided that Elliot is assessed a special fee, he has a lab. Therefore, Elliot is assessed a special fee.”

C = Elliot is a chemistry major.
P = Elliot is a physics major.
L = Elliot has a lab.
F = Elliot has to pay a special fee.

5) “Being kind is necessary but not sufficient for being good. Being kind is sufficient for being popular. Therefore, although someone can be kind without being good, if someone is good then he’s popular.”

K = Someone is kind.
G = Someone is good.
P = Someone is popular.
6) “Being kind is necessary but not sufficient for being good. Being kind is necessary for being popular. Therefore, although someone can be kind without being good, if someone is good then he’s popular.”

K = Someone is kind.
G = Someone is good.
P = Someone is popular.

III) Symbolized Arguments

1) \( (~L \& ~M) \rightarrow (F \rightarrow G), F \rightarrow ~H, ~G \rightarrow H \vdash L \lor M \)

2) \( ~P \rightarrow (A \lor B), ~Q \rightarrow (A \lor B), ~(P \& Q) A \rightarrow C, B \rightarrow D \vdash C \lor D \)

3) \( ~A \rightarrow D, C, (C \& D) \rightarrow B \vdash A \lor B \)

4) \( ~(A \& B), ~A \rightarrow C, ~B \rightarrow D, (C \lor D) \rightarrow ~(P \rightarrow Q) \vdash ~Q \)

5) \( A \rightarrow ~P, Q \rightarrow B, P \lor Q, B \rightarrow C, C \rightarrow A \vdash A \leftrightarrow B \)

6) \( ~Q \leftrightarrow ~P, L \rightarrow ~A, M \rightarrow B, (P \rightarrow Q) \rightarrow ~(~L \& ~M) \vdash A \rightarrow B \)

7) \( ~Q \rightarrow ~P, ~R \rightarrow P, (~R \rightarrow Q) \rightarrow (A \lor B), A \rightarrow C, B \rightarrow D, (L \rightarrow M) \rightarrow ~(C \lor D) \)

IV) Constructing Symbolized Arguments of Your Own

Many people enjoy constructing proofs for symbolized arguments because it’s sort of like solving a puzzle. Happily it’s relatively easy to write arguments of your own to prove. Simply follow these steps:

1. Write your conclusion.
2. Write a formula or formulas from which your conclusion can be derived.
3. Replace one or more of those formulas with a formula or formulas from which it can be derived.
4. Repeat step 3 until your argument is of desired complexity.
5. Reorder the premises, if desired.
6. Construct a proof of the argument to double-check.

For example, suppose that we take “P & Q” as our conclusion.

\( \vdash P \& Q \)
This conclusion is equivalent to the negated conjunction \("\neg(\neg P \lor \neg Q)\)" and so can be derived from it.

\[(\neg P \lor \neg Q) \vdash P \land Q\]

We can derive \("\neg(\neg P \lor \neg Q)\)" from \("(\neg P \lor \neg Q) \rightarrow R\)" and \("\neg R\), so let’s replace it with those formulas.

\[
\begin{align*}
(\neg P \lor \neg Q) \vdash P \land Q \\
(\neg P \lor \neg Q) \rightarrow R, \neg R \vdash P \land Q
\end{align*}
\]

We can derive \("\neg R\) from \("S \land \neg R\) and we can derived \("S \land \neg R\) from \("\neg(\neg S \lor R)\)" so let’s make that replacement.

\[
\begin{align*}
(\neg P \lor \neg Q) \rightarrow R, \neg R \vdash P \land Q \\
(\neg P \lor \neg Q) \rightarrow R, \neg(\neg S \lor R) \vdash P \land Q
\end{align*}
\]

Let’s reorder the premises, to make it a bit more challenging.

\[
A \land B, (\neg P \lor \neg Q) \rightarrow R, (A \land B) \leftrightarrow (\neg S \lor R) \vdash P \land Q
\]

And now we can construct the proof to double-check:

1. A \land B \hspace{2cm} P
2. (\neg P \lor \neg Q) \rightarrow R \hspace{2cm} P
3. (A \land B) \leftrightarrow (\neg S \lor R) \hspace{2cm} P - want P \land Q
4. (A \land B) \rightarrow (\neg S \lor R) \hspace{2cm} 3 \leftrightarrow O
5. (\neg S \lor R) \hspace{2cm} 1, 4 \rightarrow O
6. S \land \neg R \hspace{2cm} 5 DM
7. \neg R \hspace{2cm} 6 &O
8. (\neg P \lor \neg Q) \hspace{2cm} 2, 7 MT
9. P \land Q \hspace{2cm} 8 DM

It works! We’ve written an argument to prove.

Once you’ve written a handful of arguments, you can share them with friends or put them away to prove yourself later.
ANSWERS

I) Equivalences

1 1) “Being rich is not sufficient for being happy.”
R = Some is rich.
H = Someone is happy.

\(~(R \rightarrow H)\)

a) “Someone can be rich but not happy.” \(R \land \neg H\)
b) “Someone can be happy but not rich.” \(H \land \neg R\)
c) “It isn’t the case that either someone isn’t rich or they are happy.” \(\neg(\neg R \lor H)\)
d) “It isn’t the case that either someone is happy or they aren’t rich.” \(\neg(H \lor \neg R)\)
e) “It isn’t the case that either someone isn’t happy or they are rich.” \(\neg(\neg H \lor R)\)
f) “Not being rich is sufficient for not being happy.” \(\neg R \rightarrow \neg H\)

2 2) “Free will is not necessary for having moral responsibility.”
F = Someone has free will.
M = Someone has moral responsibility

\(\neg(M \rightarrow F)\)

a) “Someone can have free will without moral responsibility.” \(F \land \neg M\)
b) “Someone can have moral responsibility without free will.” \(M \land \neg F\)
c) “It isn’t the case that either someone doesn’t have free will or they do have moral responsibility.” \(\neg(\neg F \lor M)\)
d) “It isn’t the case that either someone doesn’t moral responsibility or they do have free will.” \(\neg(\neg M \lor F)\)
e) “Not having free will is necessary for not having moral responsibility.” \(\neg M \rightarrow \neg F\)
f) “Failing to have free will is not sufficient for failing to have moral responsibility.” \(\neg(F \rightarrow \neg M)\)

3 3) “Unless the class is full, she’ll take it.”
F = The class is full.
T = She’ll take the class.

\(\neg F \rightarrow T\)

a) “Either the class is full or she’ll take it.” \(F \lor T\)
b) “If she doesn’t take the class then it’s full.” \(\neg T \rightarrow F\)
c) “If she does take the class then it isn’t full.” \(T \rightarrow \neg F\)
d) “It’s not the case that the class isn’t full and she won’t take it.” \(\neg(F \land \neg T)\)
e) “The class is full only if she doesn’t take it.” \(F \rightarrow \neg T\)
f) “Her taking the class is not necessary for the class being full.” ~(F → T)

4) “The town is neither friendly nor affordable.”

F = The town is friendly.
A = The town is affordable.

~F & ~A

a) “The town’s being friendly is not necessary for it being affordable.” ~(A → F) = A & ~F
b) “The town’s being friendly is not necessary for it failing to be affordable.” ~(~A → F) = ~A & ~F
c) “It’s not the case that either the town is friendly or it’s affordable.” ~(F v A)
d) “Provided that the town is friendly, it isn’t affordable.” F → ~A
e) “Provided that the town is affordable, it isn’t friendly.” A → ~F
f) “It isn’t the case that unless the town is friendly, it’s affordable.” ~(~F → A)

II) English Arguments

5 1) “Morality is objective if either God exists or there is a moral law. Morality is objective only if people agree about what’s right and wrong. People don’t agree about what’s right and wrong. Therefore, neither God nor the moral law exist.”

O = Morality is objective.
G = God exists.
L = There is a moral law.
A = People agree about what’s right and wrong.

(G ∨ M) → O, O → A, ~A ⊢ ~G & ~M

1. (G ∨ M) → O
2. O → A
3. ~A
4. ~O
5. ~(G ∨ M)
6. ~G & ~M

Valid

6 2) “Morality is objective only if either God exists or there is a moral law. Morality is objective only if people agree about what’s right and wrong. People don’t agree about what’s right and wrong. Therefore, neither God nor the moral law exist.”

O = Morality is objective.
G = God exists.
L = There is a moral law.
A = People agree about what’s right and wrong.

O \to (G \lor M), O \to A, \neg A \vdash \neg G \land \neg M

1. O \to (G \lor M) \quad P
2. O \to A \quad P
3. \neg A \quad P - want \ \neg G \land \neg M
4. \neg O \quad 2, 3 \text{ MT}
5. \neg (G \lor M) \quad 1, 4 \text{ FNA - Invalid}
6. \neg G \land \neg M \quad 5 \text{ DM}

Invalid. This argument commits the Fallacy of Negating the Antecedent.

7) “Elliot is neither a chemistry major nor a physics major if he doesn’t have a lab. Eliot is either a chemistry major or a physics major. Provided that Elliot has a lab, he has to pay a special fee. Therefore, Elliot is assessed a special fee.”

C = Elliot is a chemistry major.
P = Elliot is a physics major.
L = Elliot has a lab.
F = Elliot is assessed a special fee.

\neg L \to (\neg C \land \neg P), C \lor P, L \to F \vdash F

1. \neg L \to (\neg C \land \neg P) \quad P
2. C \lor P \quad P
3. L \to F \quad P - want F
4. \neg (\neg C \land \neg P) \quad 2 \text{ DM}
5. L \quad 1, 4 \text{ MT}
6. F \quad 3, 5 \to O

Valid.

8) “Elliot is neither a chemistry major nor a physics major if he doesn’t have a lab. Eliot is either a chemistry major or a physics major. Provided that Elliot is assessed a special fee, he has a lab. Therefore, Elliot is assessed a special fee.”

C = Elliot is a chemistry major.
P = Elliot is a physics major.
L = Elliot has a lab.
F = Elliot has to pay a special fee.

\neg L \to (\neg C \land \neg P), C \lor P, F \to L \vdash F

1. \neg L \to (\neg C \land \neg P) \quad P
2. \( C \lor P \) 
3. \( F \to L \) 
4. \( \neg(C \land \neg P) \) 
5. \( L \) 
6. \( F \) 

Invalid. This argument commits the Fallacy of Assuming the Consequent.

9) “Being kind is necessary but not sufficient for being good. Being kind is sufficient for being popular. Therefore, although someone can be kind without being good, if someone is good then he’s popular.”

\( K = \text{Someone is kind.} \)
\( G = \text{Someone is good.} \)
\( P = \text{Someone is popular.} \)

\[ (G \to K) \land \neg(K \to G), K \to P \vdash (K \land \neg G) \land (G \to P) \]

1. \( (G \to K) \land \neg(K \to G) \) 
2. \( K \to P \) 
3. \( \neg(K \to G) \) 
4. \( K \land \neg G \) 
5. \( G \to K \) 
6. \( G \to P \)

Valid

10) “Being kind is necessary but not sufficient for being good. Being kind is necessary for being popular. Therefore, although someone can be kind without being good, if someone is good then he’s popular.”

\( K = \text{Someone is kind.} \)
\( G = \text{Someone is good.} \)
\( P = \text{Someone is popular.} \)

\[ (G \to K) \land \neg(K \to G), P \to K \vdash (K \land \neg G) \land (G \to P) \]

1. \( (G \to K) \land \neg(K \to G) \) 
2. \( P \to K \) 
3. \( \neg(K \to G) \) 
4. \( K \land \neg G \) 
5. \( G \to K \) 
6. \( G \to P \)
Invalid. A mistake is committed on line 6.

### III) Symbolized Arguments

#### 11) (~L & ~M) → ~(F → G), F → ~H, ~G → H ⊢ L ∨ M

1. (~L & ~M) → ~(F → G)  P
2. F → ~H  P
3. ~G → H  P - want L ∨ M (I’ll prove ~(~L & ~M) and use DM)
4. ~H → G  3 → CN
5. F → G  2, 4 CH
6. ~(~L & ~M)  1,5 MT
7. L ∨ M  6 DM

#### 12) ~P → (A ∨ B), ~Q → (A ∨ B), ~(P & Q) A → C, B → D ⊢ C ∨ D

1. ~P → (A ∨ B)  P
2. ~Q → (A ∨ B)  P
3. ~(P & Q)  P
4. A → C  P
5. B → D  P - want C ∨ D (I’ll use CD)
6. ~P ∨ ~Q  3 DM
7. A ∨ B  1, 2, 6 SD
8. C ∨ D  4, 5, 8 CD

#### 13) ~A → D, C, (C & D) → B ⊢ A ∨ B

1  1. ~A → D  P
2  2. C  P
3  3. (C & D) → B  P – want A ∨ B (I’ll prove ~A → B and use AR)
4  4. ~A  PP – want B
1,4  5. D  1,4 → O
1,2,4  6 C & D  2,5 &I
1,2,3,4  7. B  3,6 → I
1,2,3  8. ~A → B  4 – 7 → I
1,2,3  9. A ∨ B  8 AR

#### 14) ~(A & B), ~A → C, ~B → D, (C v D) → ~(P → Q) ⊢ ~Q

1. ~(A & B)  P
2. ~A → C  P
3. ~B → D  P
4. \((C \lor D) \rightarrow \neg(P \rightarrow Q)\)  \(P \rightarrow \neg Q\)
5. \(\neg A \lor \neg B\)  1 DM
6. \(C \lor D\)  2, 3, 5 CD
7. \(\neg(P \rightarrow Q)\)  4, 6 \(\rightarrow O\)
8. \(P \& \neg Q\)  7 Neg AR
9. \(\neg Q\)  8 &O

15  \(A \rightarrow \neg P, Q \rightarrow B, P \lor Q, B \rightarrow C, C \rightarrow A \vdash A \leftrightarrow B\)

1. \(A \rightarrow \neg P\)  P
2. \(Q \rightarrow B\)  P
3. \(P \lor Q\)  P
4. \(B \rightarrow C\)  P
5. \(C \rightarrow A\)  P - want \(A \leftrightarrow B\)
6. \(P \rightarrow \neg A\)  1 CN
7. \(\neg A \lor B\)  2, 3, 6 CD
8. \(A \rightarrow B\)  7 AR
9. \(B \rightarrow A\)  4, 5 CH
10. \(A \leftrightarrow B\)  8, 9 \(\leftrightarrow I\)

16  \(\neg Q \leftrightarrow \neg P, L \rightarrow \neg A, M \rightarrow B, (P \rightarrow Q) \rightarrow \neg(\neg L \& \neg M) \vdash A \rightarrow B\)

1. \(\neg Q \leftrightarrow \neg P\)  P
2. \(L \rightarrow \neg A\)  P
3. \(M \rightarrow B\)  P
4. \((P \rightarrow Q) \rightarrow \neg(\neg L \& \neg M)\)  P - want \(A \rightarrow B\)
5. \(P \leftrightarrow Q\)  1 \(\leftrightarrow CN\)
6. \(P \rightarrow Q\)  5 \(\leftrightarrow O\)
7. \(\neg(\neg L \& \neg M)\)  4, 6 \(\rightarrow O\)
8. \(L \lor M\)  7 DM
9. \(\neg A \lor B\)  2, 3, 8 CD
10. \(A \rightarrow B\)  9 AR

17  \(\neg Q \rightarrow \neg P, \neg R \rightarrow P, (\neg R \rightarrow Q) \rightarrow (A \lor B), A \rightarrow C, B \rightarrow D, (L \rightarrow M) \rightarrow \neg(C \lor D) \vdash L \& \neg M\)

1. \(\neg Q \rightarrow \neg P\)  P
2. \(\neg R \rightarrow P\)  P
3. \(\neg R \rightarrow Q\)  \(\rightarrow (A \lor B)\)  P
4. \(A \rightarrow C\)  P
5. \(B \rightarrow D\)  P
6. \((L \rightarrow M) \rightarrow \neg(C \lor D)\)  P - want \(L \& \neg M\)
7. \(P \rightarrow Q\)  1 CN
8. \(~R \rightarrow Q\) 3, 8 CH
9. \(A \lor B\) 3, 7 \(\rightarrow\)O
10. \(C \lor D\) 4, 5, 9 CD
11. \(~(L \rightarrow M)\) 6, 10 MT
12. \(L \& ~M\) 11 Neg AR

18) 8) \(A \rightarrow B, \neg A \rightarrow R, B \rightarrow R, (P \lor Q) \rightarrow \neg R, \neg (P \& S) \vdash \neg S\)

1. \(A \rightarrow B\) P
2. \(\neg A \rightarrow R\) P
3. \(B \rightarrow R\) P
4. \((P \lor Q) \rightarrow \neg R\) P
5. \(~(\neg P \& S)\) P - want \(\neg S\)
6. \(A \lor B\) 1 AR
7. \(R\) 2, 3, 6 SD
8. \(~(P \lor Q)\) 4, 7 MT
9. \(~P \& \neg Q\) 8 DM
10. \(P \lor \neg S\) 5 DM
11. \(\neg P\) 9 \&O
12. \(~S\) 10, 11 \(\lor\)O

19) 9) \(~(L \lor M), (A \& ~D) \rightarrow L, \neg A \rightarrow T, D \rightarrow \neg S, \neg M \rightarrow S \vdash T\)

1. \(~(L \lor M)\) P
2. \((A \& ~D) \rightarrow L\) P
3. \(~A \rightarrow T\) P
4. \(D \rightarrow \neg S\) P
5. \(~M \rightarrow S\) P - want \(T\)
6. \(~L \& ~M\) 1 DM
7. \(~L\) 6 \&O
8. \(~(A \& ~D)\) 2, 7 MT
9. \(~A \lor D\) 8 DM
10. \(T \lor \neg S\) 3, 4, 9 CD
11. \(~M\) 6 \&O
12. \(S\) 5, 11 \(\rightarrow\)O
13. \(T\) 10, 12 \(\lor\)O

20) 10) \(P \lor Q, P \rightarrow R, (S \& Q) \rightarrow (R \lor T), \neg(S \rightarrow T), A \rightarrow \neg R \vdash \neg A\)

1 1. \(P \lor Q\) P
2 2. \(P \rightarrow R\) P
3 3. \((S \& Q) \rightarrow (R \lor T)\) P
4  4. ~(S → T)    P
5  5. A → ~R    P - want ~A (want Q → R)
6  6. Q    PP – want R
4  7. S & ~T    4 Neg AR
4  8. S    7 &O
4,6  9. S & Q    6, 8 &I
3,4,6  10. R ∨ T    3, 9 →O
4  11. ~T    7 &O
3,4,6  12. R    10, 11 ∨O
3,4  13. Q → R    6 – 12 →I
1,2,3,4  14. R    1, 2, 13 SC
1,2,3,4,5  15. ~A    5, 14 MT