CHAPTER 2 - CONJUNCTIONS and BICONDITIONALS

Here, you’ll learn:

How to understand complex sentences by
• symbolizing conjunctions
• assessing conjunctions
• symbolizing biconditionals
• assessing biconditionals

How to assess arguments for validity by
• establishing that arguments are valid using the rules &I, &O, & COM, & ASSOC ↔I, ↔O, and ↔ COM

Symbolizing with One Connector – Part I

1) Symbolizing with “and”

Now that we’ve learned about conditionals, we’re ready to take on another connector: “and.”

Rule: “□ and ○” is symbolized “□ & ○.”
(If you have difficulty making an “&,” you may write “□ ^ ○” instead.)

A compound statement with “&” as the main connector is called a conjunction, and the statements connected by the “&” are called conjuncts.

Conjunction

[Conjunct] & [Conjunct]

Practice

Symbolize the following. The answers are in the endnotes.

1. “The earth revolves around the sun and the moon revolves around the earth.”¹

2. “The house is large and well-furnished.”²

3. “He enjoys playing soccer and baseball.”³
II) Some “and”s aren’t Conjunctions

Not every sentence with the word “and” in it is a conjunction.

For instance, consider the sentence “Joyce and Carol are sisters.” If we symbolized this sentence as “J & C” we’d be symbolizing “Joyce is a sister and Carol is a sister,” which isn’t quite what “Joyce and Carol are sisters” conveys in most conversational contexts. In most conversational contexts, “Joyce and Carol are sisters” conveys the idea that Joyce and Carol are sisters of each other, and the best we can do to symbolize that idea right now is “S.”

Here’s another example of a sentence that uses “and” but isn’t really a conjunction. Suppose that you’re at the grocery store and you hear a frustrated parent say to a misbehaving child, “You keep that up and you’ll have a ‘time out’ when we get home.” This sentence isn’t a conjunction; it’s a conditional. The parent isn’t saying “You will keep this up and you will have a ‘time out,’” (K & T) but rather “If you keep this up then you’ll have a ‘time out,’” (K → T).

So, although “and” is “&” most of the time, we always need to keep our wits about us.

Assessing Conjunctions

I) What Makes a Conjunction True

The truth table for “&” is what you’d probably expect it to be, so we can utilize our intuitions. Suppose, for example, that I tell you, “Sara’s in the living room and she’s talking on the phone.” If Sara is in the living room and talking on the phone, then I’ve told the truth. But if Sara is in the living room but not talking on the phone, or if she’s talking on the phone in some other room, or if she’s neither in the living room nor talking on the phone, then what I said was false. We can represent this in the following chart.

<table>
<thead>
<tr>
<th>Sara is in the living room</th>
<th>Sara is talking on the phone</th>
<th>Sara is in the living room and she is talking on the phone</th>
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\[footnote{1}{If you study relational predicate logic, you’ll see a somewhat more satisfying way to symbolize “Joyce and Carol are sisters,” but to really understand relational predicate logic you need to learn propositional logic first.}\]
We can generalize from this to represent the truth conditions for a conjunction in a truth table.

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<th>□ &amp; ○</th>
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<tbody>
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</tbody>
</table>

And we can summarize the truth table in the following rule:

**Rule:** A conjunction is true when and only when both conjuncts are true.

This gives us the *defining logical feature* of “&.” What *makes* an “and” an “and” is the fact that the truth of the whole “trickles down” to the truth of the parts.

In other words, if we’re presented with a conjunction,

□ & ○

and told that the conjunction is true,

T □ & ○

we can infer that both conjuncts are true as well.

T □ & ○ T T
That’s what I mean when I say that truth “trickles down” from the whole to the parts of a conjunction.

Because “truth trickling” is the crucial logical feature of a conjunction, whenever truth trickles down over a connector, that connector should be symbolized with an “&,” even if that connector isn’t the English word “and” or its direct translation.

Symbolizing with One Connector – Part II

I) Symbolizing with “and” Equivalents

Now that we know that a connector should be symbolized with “&” whenever truth trickles down over it from the whole to the parts, we can see a number of other ways to express conjunctions.

For example, suppose we know that the following sentence is true:

“The meal was delicious, moreover it was well presented.”

We know from this that “the meal was delicious” is true and we know that “the meal was well presented” is true.

ii It’s worth noting that truth does not “trickles down” over a conditional. If we know that “P → Q” is true, we don’t know that “P” is true and we don’t know that “Q” is true because “P → Q” is true under all of the following conditions: “P” and “Q” are both true; “P” if false and “Q” is true; “P” and “Q” are both false.
Truth trickles down over “moreover,” so “moreover” is symbolized as “&.

“The meal was delicious, moreover it was well presented.”
D & P

Now suppose we know that the following sentence is true:

“The day was cold, furthermore it was rainy.”

We know from this that “the day was cold” is true and we know that “the day was rainy” is true.

Truth trickles down over “furthermore,” so “furthermore” is symbolized as “&.

“The day was cold, furthermore it was rainy.”
C & R

Neither of those sentences might have surprised you because “moreover” and “furthermore” are basically synonymous with “and.” “Moreover,” “furthermore,” and “and,” all have the function of conjoining facts that harmonize fairly well with each other. The fact that the meal was delicious and the fact that the meal was well presented harmonize in a chorus of praise for the meal. The fact that the day was cold and the fact that the day was rainy harmonize in chorus of complaint.

In contrast, consider the sentence

“The day was cold, furthermore it was rainy.”

Neither of those sentences might have surprised you because “moreover” and “furthermore” are basically synonymous with “and.” “Moreover,” “furthermore,” and “and,” all have the function of conjoining facts that harmonize fairly well with each other. The fact that the meal was delicious and the fact that the meal was well presented harmonize in a chorus of praise for the meal. The fact that the day was cold and the fact that the day was rainy harmonize in chorus of complaint.

In contrast, consider the sentence

“Joe thought that he would lose, but he won first place.”

Here we have two facts that contrast with each other a little. Joe took first place despite the fact that he thought he would lose. This implied contrast is missing from the English word “and,” which is why “but” and “and” aren’t exactly synonyms.
However if we’re told that “Joe thought that Joe would lose, but he won first place” is true,

\[
\text{T}
\]

“Joe thought that he would lose, but he won first place.”

then we know that “Joe thought that he would lose” is true and we know that “Joe won first place” is true.

\[
\text{T}
\]

“Joe thought that he would lose, but he won first place.”

In other words, truth trickles down over the word “but,” so “but” is symbolized as “&.”

“Joe thought that he would lose, but he won first place.”

\[ L \land W \]

Similarly, all of the following sentences are symbolized as conjunctions:

“Joe thought that he would lose, yet he won first place.”

“Joe thought that he would lose, however he won first place.”

“Joe thought that he would lose, nevertheless he won first place.”

\[ L \land W \]

Before you practice symbolizing some conjunctions, there two more expressions we need to consider. First, let’s take a look at the sentence

“She auditioned for the play, although she suffers from stage-fright.”

As before, if we know that the whole sentence is true

\[
\text{T}
\]

“She auditioned for the play, although she suffers from stage-fright.”
then we know that the parts are true

\[
T
\]

\[
\text{“She auditioned for the play, } \textit{although} \text{ she suffers from stage-fright.”}
\]

\[
T \quad B
\]

\[
T
\]

This shows that “although” is symbolized as a conjunction

\[
\text{“She auditioned for the play, } \textit{although} \text{ she suffers from stage-fright.”}
\]

\[
A \ & \ S
\]

And identical reasoning shows us that “even though” is symbolized as a conjunction as well.

\[
T
\]

\[
\text{“She auditioned for the play, } \textit{even though} \text{ she suffers from stage-fright.”}
\]

\[
T \quad T
\]

\[
\text{“She auditioned for the play, } \textit{even though} \text{ she suffers from stage-fright.”}
\]

\[
A \ & \ S
\]

It’s important to understand that “although” and “even though” are conjunctions because both expressions can appear in the front of the sentence, instead of in the middle.

Instead of saying

\[
\text{“She auditioned for the play, } \textit{although} \text{ she suffers from stage-fright.”}
\]

we could say

\[
\text{“Although she suffers from stage-fright, she auditioned for the play.”}
\]

and it would still be symbolized

\[
A \ & \ S
\]
Instead of saying

“She auditioned for the play, even though she suffers from stage-fright.”

we could say

“Even though she suffers from stage-fright, she auditioned for the play.”

and it would still be symbolized

\[ A \land S \]

When “although” or “even though,” appear at the beginning of a sentence, we simply place the “&” where the comma appears.

One of the most common symbolization mistakes is to symbolize sentences of the form “Although P, Q” and “Even though P, Q” as “P \to Q.” This mistake almost always stems from confusing “Although” and “Even though,” with “Provided that,” because all of these expression can appear at the beginning of a sentence and because “provided that” does, in fact, give us a conditional. But don’t be deceived! “Although” and “Even though,” are very different than “Provided that.” “Although” and “Even though,” are conjunctions. “Provided that,” is a conditional.

| Although P, Q. | P \land Q |
| Even though P, Q. | P \land Q |
| Provided that P, Q. | P \to Q |

We can summarize all this as follows:

Rule: If truth “trickles down” over a connector from the whole sentence to the parts of the sentence, then that connector is a conjunction and is symbolized with “&.”

The following connectors are all conjunctions in their normal usage:

1. “and”
2. “moreover”
3. “furthermore”
4. “but”
5. “yet”
6. “however”
7. “nevertheless”
8. “although”
9. “even though”
Practice

Symbolize the following. The answers are in the endnotes.

1. “Dogs are intelligent and affectionate.”
2. “Cats are clean, moreover they’re independent.”
3. “Guinea pigs are amusing, furthermore they’re social creatures.”
4. “Tropical fish are beautiful but they’re difficult to care for.”
5. “Squirrels can be a nuisance, yet they’re fun to watch.”
6. “Parrots are clever, however they’re messy.”
7. “Horses are expensive, nevertheless Ann wants one.”
8. “Although hedgehogs are prickly, they’re incredibly cute.”
9. “Even though some people are afraid of rats, they make great pets.”
10. “Provided that you socialize your puppy well, he should be very friendly.” (Be careful here!)

Symbolizing with More than One Connector

So far, we’ve gone from English sentences to their symbolizations. Now let’s reverse the process and go from symbolizations to English sentences.

Let’s let
“The book is entertaining” be “E,”
“The book is informative” be “I,” and
“The book is worth reading” be “R.”

“(E & I) → R” would be the sentence “If the book is entertaining and informative then the book is worth reading.”

“E & (I → R)” would be the sentence “The book is entertaining, and if the book is informative then it’s worth reading.

These sentences are saying quite different things.

First, “(E & I) → R” doesn’t claim that the book actually is entertaining; it simply talks about what would result if the book were entertaining (and informative, too). “E & (I → R),” on the other hand, does claim that the book is entertaining.
Second, “(E & I) → R” asserts that whether a book is worth reading is a function of two things: whether the book is entertaining and whether the book is informative. “E & (I → R),” on the other hand, asserts that whether a book is worth reading is entirely a function of whether or not it’s informative.

“(E & I) → R” and “E & (I → R)” will function differently in proofs as well.

In particular, we can do Arrow Out on “(E & I) → R” because “→” is the main connector of this sentence. “(E & I) → R” is a conditional.

We cannot do Arrow Out on “E & (I → R)” because “→” is not the main connector of that sentence. “E & (I → R)” is a conjunction.

The moral of this story, of course, is that it matters very much where the parentheses go, so when we symbolize more complex sentences, we always need to make sure that we correctly identify the main connector, the connector the “hooks up” the entire sentence.

Practice

Symbolize the following. The answers are in the endnotes.

1. “If Hank is a lawyer then he’s both smart and assertive.”14

2. “Linda is an accountant only if she’s good with numbers, but she always miscalculates the tip.”15

3. “Being a good listener is a necessary condition for being a good doctor, and a sufficient condition for being a good friend.”16

4. “Provided that George is both a good programmer and a good communicator, he’ll get the job and he’ll rise quickly through the ranks.”17

Biconditionals

There’s a special kind of sentence that uses more than one connector. We’ll sneak up on it by considering the following sentence:

i. “We’ll go on a picnic if it’s a nice day and we’ll go on a picnic only if it’s a nice day.”

Letting “P” be “we’ll go on a picnic” and “N” be “it’s a nice day,” this sentence is symbolized like so:

i. “We’ll go on a picnic if it’s a nice day and we’ll go on a picnic only if it’s a nice day.”
(N → P) & (P → N)

Now, let’s notice that instead of saying

i. “We’ll go on a picnic if it’s a nice day and we’ll go on a picnic only if it’s a nice day.”
we could have said

ii. “We’ll go on a picnic if and only if it’s a nice day.”

and it because sentence ii is saying the same thing as sentence i, we can symbolize ii exactly as we symbolized i.

ii. “We’ll go on a picnic if and only if it’s a nice day.”
(N → P) & (P → N)

We can symbolize the sentence this way, but because “if and only if” is a relatively common expression, there’s some advantage to symbolizing it with one symbol, “↔,” like so:

ii. “We’ll go on a picnic if and only if it’s a nice day.”
P ↔ N

“P ↔ N” shows that we have two arrows, one going from N to P (the “←” part) and one going from P to N (the “→” part), and so “P ↔ N” is essentially a contraction of “(N → P) & (P → N).”

Because “if and only if,” symbolized as “↔,” indicates that there are two arrows, we’ll call sentences with “if and only if” as the main connector biconditionals. It’s important to remember, though, that a biconditional is really more of a conjunction than it is a conditional; it’s the conjunction of two conditionals.

Rule: “□ if and only if ○” is symbolized as □ ↔ ○

Such sentences are sometimes called “biconditionals.”

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iii Biconditionals pose an interesting pedagogical challenge. An instructor is faced with four choices:
1. Introduce biconditionals as a separate connector (↔) with its own proof rules (↔I, ↔O). But it’s not really a separate connector with separate rules. It’s really a conjunction of two conditionals.
2. Introduce biconditionals in the lesson on conjunctions, as (P → Q) & (Q → P), and keep it that way. But this makes symbolizing some sentences far too complex.
3. Omit biconditionals entirely. But biconditional expressions are common enough to warrant attention.
4. Introduce biconditionals in the lesson on conjunctions, as (P → Q) & (Q → P), but then transition to “↔.” This is what I’ve chosen to do.
Because a biconditional is the conjunction of two conditionals, one conditional going each way, we can discover the truth conditions for a biconditional by first determining the truth conditions for the conjunction, like this:

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and then noting that the truth conditions for the biconditional will be just the same.

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And inspection of this truth-table allows us to see that a biconditional is true when and only when both parts “match” – being both true or both false.

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We can summarize this in the following rule:

**Rule:** A biconditional is true when and only when both parts have the same truth value.

Just as there are a variety of ways to express conditionals and conjunctions, there are a variety of ways to express biconditionals.

One way to express a biconditional is to write “iff,” which is simply short for “if and only if.” “Iff” tends to be restricted to technical contexts, however. You wouldn’t want to write “iff” in a letter to the editor.

“Is necessary and sufficient for,” on the other hand, is a very common way to express a biconditional. The fact that “is necessary and sufficient for” is a biconditional makes sense when you think about it. After all, if P is necessary and sufficient for Q, then P is necessary for Q (Q → P) and sufficient for Q (P → Q), so we have P ↔ Q.
Rule: The following expressions are equivalent to “if and only if”:

1. “iff”
2. “is necessary and sufficient for”

Practice

Symbolize the following. The answers are in the endnotes.

1. “She’ll pass the class if and only if she studies.”

2. “Computers can have minds iff minds are physical.”

3. “Completing all of the degree requirements is a necessary and sufficient condition for being eligible to graduate.”

4. “Understanding is sufficient for wisdom if and only if understanding is necessary for morality.”

5. “The belief that the ethicality of an action is determined by its consequences and the belief that happiness is the ultimate goal of action are jointly a necessary and sufficient condition for the acceptance of utilitarianism.”

6. “God exists if and only if the universe is a well-ordered place, and the universe is a well-ordered place if and only if it’s amenable to mathematical description.”

PRIMATIVE INFERENCE RULES: &I (and $\leftrightarrow$I)

The &I Rule

Take a moment to consider the following argument forms.

$\square$, $\Diamond \vdash \square \& \Diamond$.

$\square$, $\Diamond \vdash \Diamond \& \square$.

Does it make sense to you that they’re valid? The idea is simply that when we put two true sentences together with an “&,” we get another true sentence.

For example,

“The earth revolves around the sun” is true, and

“the moon revolves around the earth” is true, so
"the earth revolves around the sun and the moon revolves around the earth," is true, as is "the moon revolves around the earth and the earth revolves around the sun."

We’ll take this as our second primitive inference rule, and we’ll call it “And In” (or “&I”) because it shows us how to put sentences into an “And” statement.

& I: □, □ ⊢ □ & □

As with “→O,” there are some things that we should note before we use “&I” in proofs.

Note: &I is a two line rule.

For instance, if we’re faced with this argument

A, B, C ⊢ A & (B & C)

we can’t prove it as follows, because this proof involves using &I on three lines at once.

1. A  P
2. B  P
3. C  P – want A & (B & C)
4. A & (B & C) 1, 2, 3 &I – No. This line is incorrect

Instead, we’ll need to use &I twice, first conjoining “B” and “C” in order to group them as one unit,

1. A  P
2. B  P
3. C  P – want A & (B & C)
4. B & C 2, 3 &I

and then conjoining “A” on line 1 with “B & C” on line 4.

1. A  P
2. B  P
3. C  P – want A & (B & C)
4. B & C 2, 3 &I
5. A & (B & C) 1, 4 &I
Note: Any statement, simple or compound, may be □ or O.

So, in order to prove this argument

\[ P \rightarrow R, \ Q \ & \ S, \ L \vdash [L \ & \ (P \rightarrow R)] \ & \ (Q \ & \ S) \]

we can conjoin “L” on line 3 with “P \rightarrow R” on line 1.

1. P \rightarrow R
2. Q \ & \ S
3. L \ P – want [L \ & \ (P \rightarrow R)] \ & \ (Q \ & \ S)
4. L \ & \ (P \rightarrow R) \ 1, 3 \ &I

and then we can conjoin “L \ & \ (P \rightarrow R)” on line 4 with “Q \ & \ S” on line 2.

1. P \rightarrow R
2. Q \ & \ S
3. L \ P – want [L \ & \ (P \rightarrow R)] \ & \ (Q \ & \ S)
4. L \ & \ (P \rightarrow R) \ 1, 3 \ &I
5. [L \ & \ (P \rightarrow R)] \ & \ (Q \ & \ S) \ 2, 4 \ &I

Because we can always conjoin two ideas with an “&,” we don’t really need a forward hint telling us when we can use &I. A backward hint can be helpful, however.

Backward Hint: If you want □ \ & \ O, see if you can get □ and O and use &I.

For example, suppose we’re presented with this argument

\[ A, \ (A\&B) \rightarrow (P\rightarrow R), \ P, \ B \vdash R \ & \ (P \rightarrow R) \]

1. A
2. (A\&B) \rightarrow (P\rightarrow R)
3. P
4. B

Because we want to prove “R \ & \ (P \rightarrow R)” let’s see if we can prove “R” and “P \rightarrow R.” If we can, we will be able to use &I to obtain our conclusion.

So, can we prove “P \rightarrow R?” Well, that’s the consequent of the conditional on line 2, so if we can get the antecedent of the conditional on line 2, “A \ & \ B,” we can derive “P \rightarrow R” using →O.

1. A
2. (A\&B) \rightarrow (P\rightarrow R)
3. P
4. B

P - want R \ & \ (P\rightarrow R)
Can we get “A & B?” Sure! We just need to use &I on lines 1 and 4. So let’s derive “A & B” and then get “P → R.”

1. A  
2. (A&B) → (P→R)  
3. P  
4. B  
5. A & B  
6. P → R

Now we just need to get “R.” Can we? Absolutely! “R” follows from “P” on line 3 and “P → R” on line 6.

1. A  
2. (A&B) → (P→R)  
3. P  
4. B  
5. A & B  
6. P → R  
7. R

And now, &I and we’re done!

1. A  
2. (A&B) → (P→R)  
3. P  
4. B  
5. A & B  
6. P → R  
7. R  
8. R & (P → R)

Practice

Prove the following arguments. The answers follow in the endnotes.

1. “Minds causally interact with bodies. Bodies are physical things. And if minds causally interact with bodies and bodies are physical things then minds are physical things. Therefore minds are physical things.”

Let C = “Minds causally interact with bodies,”
B = “Bodies are physical things,” and
M = “Minds are physical things.”
2. T, S, T → M, S → L |- M & L

3. R → S, R → T, (S&T) → (R → L), R |- L & R

4. [(A → B) & (C → D)] → [(G & F) → E], (A → B) → F, (C → D) → G, A → B, C → D |- E

The ↔ I Rule

Because a biconditional is simply a conjunction of conditionals, we can have a “↔ I” rule just as we have a “& I” rule. Here it is:

↔ I:  \[ \square \rightarrow O, O \rightarrow \square \vdash \square \leftrightarrow O \]

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</tr>
</tbody>
</table>

Hopefully, ↔ I makes sense to you. If it doesn’t, think about it this way:

From “\( \square \rightarrow O \)” and “\( O \rightarrow \square \)” we can derive “(\( \square \rightarrow O \)) & (\( O \rightarrow \square \))” using &I, and “(\( \square \rightarrow O \)) & (\( O \rightarrow \square \))” is equivalent to “\( \square \leftrightarrow O \)”.
So “\( \square \rightarrow O, O \rightarrow \square \vdash \square \leftrightarrow O \)”.

Similarly, from “\( \square \rightarrow O \)” and “\( O \rightarrow \square \)” we can derive “(\( O \rightarrow \square \)) & (\( \square \rightarrow O \))” using &I, and “(\( O \rightarrow \square \)) & (\( \square \rightarrow O \))” is equivalent to “\( O \leftrightarrow \square \)”.
So “\( \square \rightarrow O, O \rightarrow \square \vdash O \leftrightarrow \square \)”.

The backward hint for ↔ I is especially helpful because it tells us how to establish a biconditional: simply establish each of the “one directional” conditionals and then use ↔ I.

Backward Hint: If you want \( O \leftrightarrow \square \), see if you can get \( \square \rightarrow O \) and \( O \rightarrow \square \) and use ↔ I.

For example, consider the following argument (and if you’re feeling pretty confident, try proving it on your own and then compare your proof with mine):

L → ((A & B) → C), L, ((A & B) → C) → (C → (A & B)) |- (A & B) ↔ C

1. L → ((A & B) → C) P
2. L P
3. ((A & B) → C) → (C → (A & B)) P - want (A & B) ↔ C

Because we want to prove a biconditional, let’s see if we can prove each of the individual conditionals and then use ↔ I. I’ll note these conditionals as “sub-wants.”
1. \( L \to ((A \land B) \to C) \)  
2. \( L \)  
3. \( (A \land B) \to C \to (C \to (A \land B)) \)  
   \[ \text{P - want } (A \land B) \leftrightarrow C \]  
   \[ - \text{ want } (A \land B) \to C \]  
   \[ - \text{ want } C \to (A \land B) \]  

I can see \((A \land B) \to C\) as the consequent of the conditional on line 1, with the antecedent of that conditional on line 2. This means that we can use \(\to\)O to get \((A \land B) \to C\).

4. \( (A \land B) \to C \)  
   1, 2 \(\to\)O

And now we can use \(\to\)O on lines 3 and 4 to derive \(C \to (A \land B)\)

5. \( C \to (A \land B) \)  
   3, 4 \(\to\)O

Which allows us to use \(\leftrightarrow\)I on lines 4 and 5, to complete the proof!
Practice

Prove the following arguments. The answers follow in the endnotes.

1. “If utilitarianism is true then an action is good if it maximizes happiness and if utilitarianism is true then an action is good only if it maximizes happiness. Utilitarianism is true. Therefore an action is good if and only if it maximizes happiness.”

Symbolize each sentence as one formula, and let “U” be “utilitarianism is true,” “H” be an action maximizes happiness,” and “G” be an action is morally good.

2. $P \rightarrow Q$, $Q \rightarrow P$, $(P \leftrightarrow Q) \rightarrow R$, $(Q \rightarrow P) \rightarrow S \vdash S \land R$

3. $A \rightarrow (B \rightarrow C)$, $A$, $(A \land D) \rightarrow (C \rightarrow B)$, $D \vdash C \leftrightarrow B$

4. $H, H \rightarrow M$, $(H \land M) \rightarrow L$, $[(H \rightarrow M) \land L] \rightarrow (M \rightarrow H) \vdash M \leftrightarrow H$

5. $B \rightarrow C$, $A$, $A \rightarrow B$, $(A \land C) \rightarrow (P \rightarrow Q)$, $(B \rightarrow C) \rightarrow (Q \rightarrow P) \vdash P \leftrightarrow Q$

6. $(A \leftrightarrow B) \rightarrow (E \rightarrow C)$, $A \rightarrow B$, $L, M$, $(L \land M) \rightarrow (B \rightarrow A)$, $C \rightarrow E$, $(C \leftrightarrow E) \rightarrow R \vdash R$

**PRIMATIVE INFERENCE RULES: &O (and ↔ O)**

**The &O Rule**

Ready for another inference rule? Take a moment to consider the following arguments forms.

\[ \square \land \bigcirc \vdash \square \]
\[ \square \land \bigcirc \vdash \bigcirc \]

Do you see that both of these argument forms are valid? They tell us that, given a conjunction, we can infer either conjunct, and this is quite correct because (as we’ve seen when we discussed symbolizing conjunctions) if a conjunction is true then both of the conjuncts must be true as well; the truth “trickles down.”

Consequently, from

“The earth revolves around the sun and the moon revolves around the earth.”

we may infer

“The earth revolves around the sun.”
and we may infer

“The moon revolves around the earth.”

We’ll take this as our second primitive inference rule, and we’ll call it “And Out” (or “&O”) because it shows us how to take sentences out of an “And” statement.

\[
& \& \quad \&\quad \&\quad \|\quad \&\quad \|
\]

Here are some points worth noting before we use &O in a proof:

Note: This is a one line rule.

A single conjunction is all we need to infer that either of the conjuncts is true. Consequently, the following proof of the argument “A, A & D ⊨ D” is incorrect (and probably stems from thinking about the “&” as if it were an “→”).

1. A  P
2. A & D  P – want D
3. D  1, 2 &O (No! This justification is mistaken. 😞)

We can correct this proof very, easily, however, by simply removing “1” from the justification column of line 3.

1. A  P
2. A & D  P – want D
3. D  2 &O (Yes! This justification is perfect. 😊)

Of course, this means that the argument has an extra premise because line 1 isn’t used to complete the proof. That can happen – although I won’t include extra premises in the arguments that I give you (at least not intentionally).

Note: Any statement, simple or compound, may be □ or ○.

So, we can easily prove the argument “(P → Q) & (A → B) ⊨ P → Q,” as follows:

1. (P → Q) & (A → B)  P - want P → Q
2. P → Q  1 &O

Note: You can apply &O only to the main connector! (This is the case for all the “out” rules.)
To see why it’s so important that &O be applied only to the main connector, consider the following sentence:

“If Linda has the time and the money then she’ll visit Spain.” = (T & M) → S

It does not follow from this sentence that Linda actually does have the time (T) because “(T & M) → S” doesn’t say that Linda has the time. It only talks about what she’ll do if she has the time (and the money). In other words, the following argument is invalid.

(T & M) → S |- T

There had better, then, be something wrong with the following proof

1. (T & M) → S  P
2. T  1 &O

There had better be something wrong with this proof because if this proof is correct then we can construct a correct proof for an invalid argument. If we can do that then we can’t conclude that an argument is valid from the fact that we can prove it and this would spell disaster because assessing arguments for validity is the whole reason for constructing proofs in the first place. Luckily, there is something wrong with this proof; the application of &O is incorrect because “&” isn’t the main connector in line 1.

1. (T & M) → S  P
2. T  1 &O (No! This inference is mistaken. 😞)

Similarly, it “If Linda has the money then she’ll visit Spain” does not follow from “If Linda has the time and the money then she’ll visit Spain” because Linda could have the money but not the time to go. In other words, the following argument is invalid.

(T & M) → S |- M → S

There had better, then, be something wrong with the following proof:

1. (T & M) → S  P
2. M → S  1 &O

And, happily, there is something wrong with the proof. It tries to use “&O” on line 1, but “&” isn’t the main connector in line 1.

1. (T & M) → S  P
2. M → S  1 &O (No! This inference is mistaken. 😞)
Forward Hint: If you have a line of the form □ & ○, where □ and ○ are somewhat complex, use &O. This will simplify the ideas for you and make connections between those ideas easier to see.

Suppose, for example, that we’re faced with the following argument:

(P & L) & {[L → (P → R)] & (R → S)} ⊢ S

This argument has a fairly complex premise, but we can simplify it using &O. Because &O can be used on only the main connector, however, we can use &O only the “&” that I’ve shaded yellow below. (The pink “&” and the blue “&” aren’t the main connector on line 1.) Let’s use &O on that line 1 twice, in order to “liberate” both of the conjuncts.

1. (P & L) & {[L → (P → R)] & (R → S)} P – want S
2. P & L 1 &O
3. [L → (P → R)] & (R → S) 1 &O

Now we can do &O again, twice on line 2,

4. P 2 & O
5. L 2 & O
6. L → (P → R) 3 & O
7. R → S 3 & O

At this point, all of the information contained in our original premise has been distributed to lines 4, 5, 6 and 7, which means that we don’t need to worry about lines 1, 2 and 3 anymore.

1. (P & L) & {[L → (P → R)] & (R → S)} P – want S
2. P & L 1 &O
3. [L → (P → R)] & (R → S) 1 &O
4. P 2 & O
5. L 2 & O
6. L → (P → R) 3 & O
7. R → S 3 & O
Can you complete the proof mentally from here? First we’ll do →O on lines 5 and 6, which will give us P→R. Then, from P→R and from P (on line 4), we can get R with →O. And finally, we can use →O on R and R→S to get to S. If you don’t see that mentally right away, take a minute to think it through.

Written down, the proof looks like this:

1. (P & L) & {[L → (P → R)] & (R → S)}  P – want S  
2. P & L 1 &O  
3. [L → (P → R)] & (R → S) 1 &O  
4. P 2 & O  
5. L 2 & O  
6. L → (P → R) 3 & O  
7. R → S 3 & O  
8. P → R 5, 6 →O  
9. R 4, 8 →O  
10. S 7, 9 →O

Practice

Prove the following arguments valid. The answers follow in the endnotes.

1. “Scapegoating producing the greatest happiness for the greatest number of people is sufficient for scapegoating being morally correct, and if scapegoating is morally correct then it should be legal. Scapegoating harms a small number of people. Scapegoating makes large numbers of people happy. Scapegoating does produce the greatest happiness for the greatest number of people, if it harms a small number of people while making large numbers of people happy. Therefore, scapegoating should be legal.”

Let
H = Scapegoating produces the greatest happiness for the greatest number of people.
C = Scapegoating is morally correct.
L = Scapegoating should be legal.
S = Scapegoating harms a small number of people.
N = Scapegoating makes large numbers of people happy.

2. S → (Q → T), S & Q |- T & Q  
3. Q & (P & S), S → W, (W & U) → (R → T), P → U |- (R → T) & (Q & P)  
4. R→ V, [P→ (R & S)] & L, L → (Q & P), S → W |- W & V  
5. L → (L → (L → M)), (R → U) & (U → P), L & R, M → A, (P & A) → (M → D) |- D
The ↔O Rule

Because a biconditional is simply a conjunction of conditionals, we can have a “↔O” rule much as we have a “&I” rule. Here it is:

\[
\begin{align*}
\leftrightarrow O: & \quad \Box \leftrightarrow \Diamond \quad \vdash \Box \rightarrow \Diamond \\
& \quad \Diamond \leftrightarrow \Box \quad \vdash \Diamond \rightarrow \Box
\end{align*}
\]

Does ↔O makes sense to you? If it doesn’t, think about it this way:

“\(\Box \leftrightarrow \Diamond\)” is equivalent to “(\(\Box \rightarrow \Diamond\) & (\(\Diamond \rightarrow \Box\))\),” and from “(\(\Box \rightarrow \Diamond\) & (\(\Diamond \rightarrow \Box\))” we may derive “\(\Box \rightarrow \Diamond\)” using &O.
So, from “\(\Box \leftrightarrow \Diamond\)” we may derive “\(\Box \rightarrow \Diamond\)” using ↔O.

Similarly, “\(\Diamond \leftrightarrow \Box\)” is equivalent to “(\(\Diamond \rightarrow \Box\) & (\(\Box \rightarrow \Diamond\)),” and from “(\(\Diamond \rightarrow \Box\) & (\(\Box \rightarrow \Diamond\))” we may derive “\(\Diamond \rightarrow \Box\)” using &O.
So, from “\(\Diamond \leftrightarrow \Box\)” we may derive “\(\Diamond \rightarrow \Box\)” using ↔O.

Everything we noted about &O applies to ↔O as well: 1) it’s a one-line rule, 2) any statement, simple or compound, may be \(\Box\) or \(\Diamond\), and 3) we can apply ↔O only to the main connector.

Before we put ↔O into practice, however, we should be careful to avoid two common errors that are unique to this inference rule:

First, we should be careful to avoid the following:

1. \(A \leftrightarrow B\) \(\quad\) P
2. \(A\) \(\quad\) 1 \(\leftrightarrow O\)
3. \(B\) \(\quad\) 1 \(\leftrightarrow O\)

No! Lines 2 and 3 are mistaken.

Can you see why this is wrong? And can you see why this is tempting?

It’s wrong because from a line of the form “\(\Box \leftrightarrow \Diamond\)” we can’t validly infer the two parts. “Hannah will pass the exam if and only if she studies,” may be true, for example, but “Hannah will pass the exam” doesn’t follow from that, and neither does “Hannah studies.”

Nonetheless, this inference is tempting because a biconditional really is just a conjunction. People who fall into this mistake are simply confused about what conjunction any given biconditional is and they start thinking that “\(\Box \leftrightarrow \Diamond\)” is simply the conjunction “\(\Box \& \Diamond\).” And, of course, if “\(\Box \leftrightarrow \Diamond\)” were simply the conjunction “\(\Box \& \Diamond\),” then we could infer “\(\Box\)” and “\(\Diamond\)” from it. But “\(\Box \leftrightarrow \Diamond\)” isn’t simply the conjunction “\(\Box \& \Diamond\).” It’s the conjunction “(\(\Box \rightarrow \Diamond\) & (\(\Diamond \rightarrow \Box\)).”
The second common error is a little less serious because (unlike the other error) the argument in which this error occurs is valid; the proof of it simply applies the rules incorrectly. Here it is:

1. \( A \leftrightarrow B \)  
2. \( A \)  
3. \( B \) 

\[ \rightarrow \text{O} \]

Can you see what’s wrong here? And can you see why it’s easy to fall into this error?

This proof is incorrect because \( \leftrightarrow \text{O} \) always gives us one of the two conditionals that comprise the biconditional. From “\( A \leftrightarrow B \)” we can infer “\( A \rightarrow B \)” using \( \leftrightarrow \text{O} \), and we can infer “\( B \rightarrow A \)” using \( \leftrightarrow \text{O} \). Line 3 gives us neither of these conditionals. Additionally, \( \leftrightarrow \text{O} \) is a one line rule, and line 3 is using it as if it were a two line rule.

It’s easy to fall into this error, though, because it’s easy to treat the biconditional exactly as if it were one of the conditionals that compose it. If “\( \square \leftrightarrow \circ \)” were the same as “\( \square \rightarrow \circ \)” then, of course, from “\( \square \leftrightarrow \circ \)” and “\( \square \)” we could infer “\( \circ \)” . But “\( \square \leftrightarrow \circ \)” isn’t exactly the same as “\( \square \rightarrow \circ \).” In order to correctly prove this argument, we’d need to revise the proof as follows:

1. \( A \leftrightarrow B \)  
2. \( A \)  
3. \( A \rightarrow B \) 
4. \( B \)

\[ \rightarrow \text{O} \]

Practice

Prove the following arguments valid. The answers follow in the endnotes.

1. “Free will is a necessary condition for moral responsibility, and we have free will if and only if not all causes are mechanistic ones. Some of our moral judgments are correct only if we have moral responsibility. In fact some of our moral judgments are correct. Therefore not all causes are mechanistic ones.”

Let

\( R = \) We have moral responsibility.  
\( F = \) We have free will.  
\( N = \) Not all causes are mechanistic ones.  
\( C = \) Some of our moral judgments are correct.

2. \((A \& B) \leftrightarrow (C \& D), C, D \vdash A\)

3. \( P \leftrightarrow Q, Q, P \rightarrow (R \leftrightarrow S), R \vdash S\)
So far, we’ve taken all of our inference rules as primitive; we’ve justified them by appeal to our intuitions because a bit of reflection has allowed us to see that each of these inferences is valid. If we’re in a particularly precise mood, we can justify these primitive inference rules by constructing truth tables as well; we simply construct a truth table for the argument that represents the inference form and we note in that every line on which all the premises true, the conclusion is true as well. We have not, however, derived any of our rules by appealing to a proof that employs other inference rules already acknowledged to be valid. That’s about to change.

**& COM**

The first inference rule that we’ll formally derive is “&Commutativity” or “& COM” for short. This rule tells us that “□ & ○” is equivalent to “○ & □,” that it doesn’t matter in which order the conjuncts appear in a conjunction.

You might not find the commutativity of “&” to be very surprising. After all, it doesn’t matter whether we say “The earth revolves around the sun and the moon revolves around the earth,” or “The moon revolves around the earth and the earth revolves around the sun” and instead of saying “I’ll bring the soda and you bring the chips,” we could just as well say “You bring the chips and I’ll bring the soda.”

Given the intuitive correctness of “& COM,” then, why don’t we just take it as a primitive rule? The answer is that although “Fairly easy to see as valid” is a necessary condition for being a primitive rule, it is not a sufficient condition. In other words, if we take an inference as primitive then that inference must be fairly easy to see as valid (generally speaking), but it isn’t the case that every inference that we can easily see as valid will be taken as primitive. “& COM” is a relatively “easy” inference rule that we’ll derive instead. And we’ll derive it because logicians generally try to take as few primitive inference rules as possible. There’s more “elegant simplicity” to a system that takes two primitive inference rules and derives the rest than there is to a system that takes 27 inference rules as primitive.

So, we want to prove that “□ & ○” is equivalent to “○ & □.” Using the three-lined equal sign that logicians employ to indicate equivalence, we want to prove “□ & ○ ≡ ○ & □.”

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iv Strictly speaking, this isn’t quite correct. Because we introduced “≡” as a contracted way of expressing the conjunction of two conditionals, the rules “≡I” and “≡O” were explained by referring to how we can derive a conjunction of two conditionals and what we can derive from the conjunction of two conditionals. Nonetheless, because we didn’t construct formal proofs to justify these rules, we took them as primitive in a loose sense.
As a general rule, in order to prove that two formulas are equivalent to each other, we just need to show that we can prove each of them from the other one.

\[ F_1 \equiv F_2. \]

So, in order to establish \( \square \& \bigcirc \equiv \bigcirc \& \square \), we need to complete the following two proofs:

1. \( \square \& \bigcirc \vdash \bigcirc \& \square \)
2. \( \bigcirc \& \square \vdash \square \& \bigcirc \)

And of course we can’t use \( \& \text{COM} \) in these proofs because we’re constructing these proofs in order to get \( \& \text{COM} \).

Can you see how these proofs can be completed? Take a moment to see if you can complete each proof mentally.

Here’s how I would do these proofs:

1. \( \square \& \bigcirc \) P - want \( \bigcirc \& \square \)
2. \( \square \) 1 &O
3. \( \bigcirc \) 1 &O
4. \( \bigcirc \& \square \) 2, 3 &I

1. \( \bigcirc \& \square \) P – want \( \square \& \bigcirc \)
2. \( \bigcirc \) 1 &O
3. \( \square \) 1 &O
4. \( \square \& \bigcirc \) 2, 3 &I

We’ve now proven \( \square \& \bigcirc \equiv \bigcirc \& \square \), which we can take as our first derived inference rule.

\[ \& \text{COM}: \quad \square \& \bigcirc \equiv \bigcirc \& \square \]

\& COM tells us that two expressions are equivalent to each other, which allows us to infer one expression from the other expression. For example, we can observe \& COM at work in the (remarkably short!) proof of the following argument:
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M & L |- L & M

1. M & L       - want L & M
2. L & M       1 & COM

As with all inference rules, any statement, simple or compound, may be the “□” and the “○.” For instance, we may easily use &COM to prove the following argument:

(A → B) & (C → D) |- (C → D) & (A → B)

1. (A → B) & (C → D)  P - want (C → D) & (A → B)
2. (C → D) & (A → B) 1 & COM

& COM differs from every other rule that we’ve seen so far, however, and this difference requires special note.

Note: & COM tells us that two sentences are equivalent to each other; it’s an equivalence rule. Equivalence rules are very useful because they help us to understand complex sentences by enabling us to re-write those sentences in other, equivalent, ways. Additionally, they are replacement rules. Replacement rules differ from other inference rules in two respects: 1) they go in both directions and 2) they may be used on parts of lines. In other words, they can be applied to a “non main” connector.

For example, we can use &COM to prove the following argument:  (P & Q) → R |- (Q & P) → R

1. (P & Q) → R       P – want (Q & P) → R
2. (Q & P) → R       1 & COM

Can you see how we did & COM on only the antecedent of line 1?

1. (P & Q) → R       P – want (Q & P) → R
2. (Q & P) → R       1 & COM

This is perfectly fine. & COM tells us that “P & Q” and “Q & P” are equivalent. Equivalent formulas are like synonymous words, and just like we can replace a word with its synonym in a sentence (going from “Lisa is a lawyer” to “Lisa is an attorney”) we can replace a formula with its equivalent in a larger formula (going from “(P & Q) → R” to “(Q & P) → R”).
Question: Could we have “→ COM” too?

Answer: No. “→” isn’t commutative. “P → Q” is not equivalent to “Q → P” because “P → Q” can be true and “Q → P” can be false at the same time. “If Max is a poodle then Max is a dog,” for instance, is true, but “If Max is a dog then Max is a poodle” is false. If we had “→ COM,” we would be able to construct a proof for the invalid argument “If Max is a poodle then Max is a dog. Therefore, if Max is a dog then Max is a poodle.” That would be very bad.

Practice

Complete the following proofs by filling in the blanks. The answers follow in the endnotes.

1. (A & B) → (P & Q), B & A ⊢ Q & P

   1. (A & B) → (P & Q) - P
   2. B & A - P - want Q & P
   3. ______ - 2 & COM
   4. ______ - 1, 3 → O
   5. ______ - 4 & COM

2. (A & B) → (P & Q), B & A ⊢ Q & P

   1. (A & B) → (P & Q) - P
   2. B & A - P - want Q & P
   3. (B & A) → (P & Q) - ______
   4. (B & A) → (Q & P) - ______
   5. ______ - 2, 4 → O

& ASSOC

Our second derived inference rule will be “& Associativity” or “& ASSOC” for sort. This rule tells us that “□ (□ & △)” is equivalent to “□ (□ & △),” that if we have a string of formulas conjoined together, it doesn’t matter in which two of the conjuncts we conjoin first.

As we’ve learned, in order to establish that “□ (□ & △) □ (□ & △),” we’ll need to complete the following two proofs:

   1. □ (□ & △) ⊢ (□ & △) & △
   2. (□ & △) & △ ⊢ □ & (□ & △)

(And of course we can’t use & ASSOC in these proofs because we’re constructing these proofs to establish & ASSOC.)

Take a minute to see if you can think through each of these proofs.

---

* It would be very bad because we want to be able to conclude that an argument is valid from the fact that we can construct a proof for it. Consequently, if we can construct a proof for an invalid argument then the entire reason for symbolic logic would be undercut.
Here’s how I would complete them:

1. $\square \& (O \& \triangle)$  P - want $\square \& (O \& \triangle)$
2. $\square$ 1 &O
3. $O \& \triangle$ 1 &O
4. $O$ 3 &O
5. $\triangle$ 3 &O
6. $\square \& O$ 2, 4 &I
7. $(\square \& O) \& \triangle$ 5, 6 &I

1. $(\square \& O) \& \triangle$ P – want $\square \& (O \& \triangle)$
2. $\square \& O$ 1 &O
3. $\square$ 2 &O
4. $O$ 2 &O
5. $\triangle$ 1 &O
6. $O \& \triangle$ 4, 5 &I
7. $\square \& (O \& \triangle)$ 3, 6 &I

We’ve now proven “$\square \& (O \& \triangle) \equiv (\square \& O) \& \triangle$,” so we have our next derived inference rule.

Here we have another derived replacement rule:

$$\& \text{ASSOC: }\left(\square \& (O \& \triangle)\right) \equiv (\square \& O) \& \triangle$$

Like & COM, & ASSOC tells us that two expressions are equivalent and so it enables us to prove one from the other. We can see how & ASSOC works by inspecting the proof of this following argument.

$A \& (B \& C) \vdash (A \& B) \& C$

1. $A \& (B \& C)$  P – want $(A \& B) \& C$
2. $(A \& B) \& C$ 1 & ASSOC

As always, any statement, simple or compound, may be “$\square,” “O,” and “$\triangle,” so we can use & ASSOC to prove the following argument:

$$[(R \rightarrow S) \& (P \rightarrow Q)] \& (L \rightarrow M) \vdash (R \rightarrow S) \& [(P \rightarrow Q) \& (L \rightarrow M)]$$

1. $[(R \rightarrow S) \& (P \rightarrow Q)] \& (L \rightarrow M)$  P – want $(R \rightarrow S) \& [(P \rightarrow Q) \& (L \rightarrow M)]$
2. $(R \rightarrow S) \& [(P \rightarrow Q) \& (L \rightarrow M)]$ 1 & ASSOC

And finally, because & ASSOC (like & COM) tells us that two sentences are equivalent to each other, it is a replacement rule that can be used on parts of lines.
For instance, we can use & ASSOC to prove the following argument:
P → [Q & (R & S)] ⊢ P → [(Q & R) & S]

1. P → [Q & (R & S)]       P – want P → [(Q & R) & S]
2. P → [(Q & R) & S]       1 & ASSOC

The proof employs & ASSOC on the consequent of line 1, but this is just fine. As a replacement rule, & ASSOC isn’t restricted to the main connector.

Practice

Complete the following proofs by filling in the blanks. The answers follow in the endnotes.

1. [(A & B) & C] → [R & (S & T)], A & (B & C) ⊢ (R & S) & T

        1. [(A & B) & C] → [R & (S & T)]       P
        2. A & (B & C)                                        P – want (R & S) & T
        3. ______________________ 2 & ASSOC
        4. ______________________ 1, 3 →O
        5. ______________________ 4 & ASSOC

2. [(A & B) & C] → [R & (S & T)], A & (B & C) ⊢ (R & S) & T

        1. [(A & B) & C] → [R & (S & T)]       P
        2. A & (B & C)                                        P – want (R & S) & T
        3. [(A & B) & C] → [(R & S) & T]       ______________________
        4. [A & (B & C)] → [(R & S) & T]       ______________________
        5. ______________________ 2, 4 →O
& COM and & ASSOC

We now have two derived rules: & COM, which tells us that $\land \land \equiv \equiv \equiv \equiv \land \land$, and & ASSOC, which tells us that $\land (\land \land) \equiv (\land \land) \land \land$. It can be easy to get confused about which rule is & COM and which is & ASSOC, until you know a little trick.

In & COM (short for “& COMMUTATIVITY”), the expressions actually change places, like cars change places when their drivers are commuting from one location to another.

\[ \begin{array}{cccc}
\square & \bigcirc & \equiv & \bigcirc & \square \\
\end{array} \]

In & ASSOC (short for “& ASSOCIATIVITY”), the expressions don’t change places; they simply change groupings, like guests at a dinner party who remain in their seats but change their associations, with the middle guest first talking to the person on one side of her, and then turning to converse with the person on the other side.

\[ \begin{array}{cccc}
\square & (\bigcirc \bigcirc \bigcirc \bigcirc) & \equiv & (\square \bigcirc \bigcirc \bigcirc) \\
\end{array} \]

When the terms in a conjunction both change places and alter their affiliations, & COM and & ASSOC are being used.
Practice

Complete the following proofs by filling in the blanks. The answers follow in the endnotes.

1. \((P \& Q) \& R \vdash Q \& (P \& R)\)

1. \((P \& Q) \& R\) P – want \(Q \& (P \& R)\)
2. \((Q \& P) \& R\)
3. \(Q \& (P \& R)\)

2. \((R \& S) \& T, (T \& R) \rightarrow (Q \& (P \& L)) \vdash Q \& L\)

1. \((R \& S) \& T\) P
2. \((T \& R) \rightarrow (Q \& (P \& L))\) P – want \(Q \& L\)
3. \(T \& (R \& S)\)
4. \((T \& R) \& S\)
5. \(Q \& (P \& L)\) 2, 5 \(\rightarrow\)
6. \(Q \& (L \& P)\)
7. \(Q \& (L \& P)\)
8. \((Q \& L) \& P\)
9. \(Q \& (L \& P)\)

\(\leftrightarrow\) COM and \(\leftrightarrow\) ASSOC

Can we have \(\leftrightarrow\) COMMUTATIVITY (\(\leftrightarrow\) COM) and \(\leftrightarrow\) ASSOCIATIVITY (\(\leftrightarrow\) ASSOC), as we have \& COM and \& ASSOC? Let’s see.

We’ll start with \(\leftrightarrow\) COM. Can we prove that \(\Box \leftrightarrow \bigcirc \equiv \bigcirc \leftrightarrow \Box\)? In other words, can we construct proofs for the following two arguments?

1. \(\Box \leftrightarrow \bigcirc \vdash \bigcirc \leftrightarrow \Box\)
2. \(\bigcirc \leftrightarrow \Box \vdash \Box \leftrightarrow \bigcirc\)

I think so. Take a few moments to see if you can envision or complete each proof.

Here’s how I would prove those arguments:

1. \(\Box \leftrightarrow \bigcirc \vdash \bigcirc \leftrightarrow \Box\)

1. \(\Box \leftrightarrow \bigcirc\) P - want \(\bigcirc \leftrightarrow \Box\)
2. \(\bigcirc \rightarrow \Box\) 1 \(\leftrightarrow\)O
3. \(\Box \rightarrow \bigcirc\) 1 \(\leftrightarrow\)O
4. \(\bigcirc \leftrightarrow \Box\) 2, 3 \(\leftrightarrow\)I
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2. $\circ \leftrightarrow \square \perp \square \leftrightarrow \circ$

1. $\circ \leftrightarrow \square$ P - want $\square \leftrightarrow \circ$
2. $\circ \rightarrow \square$ 1 $\leftrightarrow \bigcirc$
3. $\square \rightarrow \circ$ 1 $\leftrightarrow \bigcirc$
4. $\square \leftrightarrow \circ$ 2, 3 $\leftrightarrow \bigcirc$

And this gives us another derived inference rule: $\leftrightarrow \text{COM}$.

$\leftrightarrow \text{COM}$: $\square \leftrightarrow \circ \equiv \circ \leftrightarrow \square$

Because $\leftrightarrow \text{COM}$, like & COM, shows that two sentences are equivalent to each other, it gives us a replacement rule that can be used on parts of lines.

Now, what about $\leftrightarrow \text{ASSOC}$? Can we prove that $\square \leftrightarrow (\circ \leftrightarrow \bigtriangleup) \equiv (\square \leftrightarrow \circ) \leftrightarrow \bigtriangleup$? In other words, can we construct proofs for the following two arguments?

1. $\square \leftrightarrow (\circ \leftrightarrow \bigtriangleup) \perp (\square \leftrightarrow \circ) \leftrightarrow \bigtriangleup$
2. $(\square \leftrightarrow \circ) \leftrightarrow \bigtriangleup \perp (\square \leftrightarrow (\circ \leftrightarrow \bigtriangleup))$

I can’t think of a way to prove either argument given the inference set that we have at the moment, but that doesn’t mean that those arguments aren’t valid. In fact, they are valid and so the equivalence holds.

How do I know that the equivalence holds? I know because I’ve constructed the truth table.

First, I fill in all possible combinations of truth values for $\square$, $\circ$, and $\bigtriangleup$.

<table>
<thead>
<tr>
<th>$\square$</th>
<th>$\circ$</th>
<th>$\bigtriangleup$</th>
<th>$\square \leftrightarrow (\circ \leftrightarrow \bigtriangleup)$</th>
<th>$(\square \leftrightarrow \circ) \leftrightarrow \bigtriangleup$</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>t</td>
<td>t</td>
<td>t</td>
<td>t</td>
</tr>
<tr>
<td>t</td>
<td>t</td>
<td>f</td>
<td>t</td>
<td>t</td>
</tr>
<tr>
<td>t</td>
<td>f</td>
<td>t</td>
<td>f</td>
<td>t</td>
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<tr>
<td>t</td>
<td>f</td>
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<td>t</td>
<td>f</td>
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<tr>
<td>f</td>
<td>f</td>
<td>f</td>
<td>f</td>
<td>f</td>
</tr>
</tbody>
</table>
Then, I paste those truth values into □, ○, and △ in the formulas.

Next, I figure the truth value for the each “sub formula” in parentheses, remembering that a biconditional is true if and only if the truth values on both sides of the biconditionals match.

And finally, I figure the truth value for the each of the main connectors.
Inspecting this truth table, we can see that the truth values for each of formulas are the same. They match under every possible assignment of truth values to their components.

\[
\begin{array}{cccc|c|c}
\hline
\square & \bigcirc & \triangle & \square & (\bigcirc \leftrightarrow \triangle) & (\square \leftrightarrow \bigcirc) & (\square \leftrightarrow \triangle) \\
\hline
\text{t} & \text{t} & \text{t} & \text{t} & \text{T} & \text{T} \\
\text{t} & \text{t} & \text{f} & \text{f} & \text{F} & \text{F} \\
\text{t} & \text{f} & \text{t} & \text{t} & \text{F} & \text{F} \\
\text{t} & \text{f} & \text{f} & \text{f} & \text{T} & \text{T} \\
\text{f} & \text{t} & \text{t} & \text{t} & \text{F} & \text{F} \\
\text{f} & \text{t} & \text{f} & \text{f} & \text{T} & \text{T} \\
\text{f} & \text{f} & \text{t} & \text{t} & \text{T} & \text{T} \\
\text{f} & \text{f} & \text{f} & \text{f} & \text{F} & \text{F} \\
\hline
\end{array}
\]

This tells us that \( \square \leftrightarrow (\bigcirc \leftrightarrow \triangle) \) is equivalent to \( (\square \leftrightarrow \bigcirc) \leftrightarrow \triangle \). In other words, we’ve shown that \( \square \leftrightarrow (\bigcirc \leftrightarrow \triangle) \equiv (\square \leftrightarrow \bigcirc) \leftrightarrow \triangle \), and so we could have an \( \leftrightarrow \) ASSOC rule if we wanted one.

Do we want one? I don’t think so. Because \( \leftrightarrow \) ASSOC would be of almost no use when evaluating natural language arguments, and because I don’t want to end up with a cluttered set of inference rules, we won’t be taking \( \leftrightarrow \) ASSOC as rule. We simply won’t need it.

Practice

Complete the following proofs by filling in the blanks. The answers follow in the endnotes.

1. \((A \leftrightarrow B) \rightarrow (C \leftrightarrow D), B \leftrightarrow A, \vdash D \rightarrow C^{48}\)

   1. \((A \leftrightarrow B) \rightarrow (C \leftrightarrow D)\)  \(\vdash\)  \(\bigcirc\)
   2. \(B \leftrightarrow A\)  \(\vdash\)  \(C \leftrightarrow \bigcirc\)
   3. __________  \(\vdash\)  \(2 \leftrightarrow \text{COM}\)
   4. \(C \leftrightarrow D\)  \(\vdash\)  __________
   5. __________  \(\vdash\)  __________

2. \((A \leftrightarrow B) \rightarrow (C \leftrightarrow D), B \leftrightarrow A, \vdash D \rightarrow C^{49}\)

   1. \((A \leftrightarrow B) \rightarrow (C \leftrightarrow D)\)  \(\vdash\)  \(\bigcirc\)
   2. \(B \leftrightarrow A\)  \(\vdash\)  \(D \leftrightarrow C\)
   3. \((B \leftrightarrow A) \rightarrow (C \leftrightarrow D)\)  \(\vdash\)  __________
   4. \((B \leftrightarrow A) \rightarrow (D \leftrightarrow C)\)  \(\vdash\)  __________
   5. __________  \(\vdash\)  __________
Symbolize and prove the following arguments, given my sentence letters. The answers are in the endnotes.

3. “Francis will be convicted only if Pat tells the truth, provided that there were no witnesses to the crime and Francis left no physical evidence. Francis is smart and the crime occurred on a dark night. If the crime occurred on a dark night and Francis is smart then there were no witnesses to the crime and Francis left no physical evidence. If the police will know that Francis’s alibi is false only if Pat tells the truth then Pat’s telling the truth is sufficient for Francis’s conviction. In fact, for the police to know that Francis’s alibi is false it’s necessary that Pat tell the truth. Therefore, Pat’s telling the truth is necessary and sufficient for Francis’s conviction."

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P = Pat tells the truth.
C = Francis is convicted.
N = There were no witnesses to the crime.
L = Francis left no physical evidence.
S = Francis is smart.
D = The crime occurred on a dark night.
F = The police will know that Francis’s alibi is false

4. “Provided that Alma and Brad were together on Tuesday, Alma is guilty if and only if Brad is. If Alma and Brad were seen leaving the play together then they must have been together on Tuesday. Not only were Alma and Brad seen leaving the play together, but Alma’s fingerprints were on the safe whereas Brad always wears gloves. Alma’s fingerprints being on the safe is sufficient for her guilt and conviction. Should Brad always wear gloves, he’ll go free. Therefore, even though Brad is guilty, he’ll go free.”

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T = Alma and Brad were together on Tuesday.
A = Alma is guilty.
B = Brad is guilty.
L = Alma and Brad were seen leaving the play together.
D = Alma’s fingerprints were on the safe.
G = Brad always wears gloves.
C = Alma will be convicted.
F = Brad will go free.

Prove the following arguments. The answers are in the endnotes.

5. $(P \leftrightarrow Q) \leftrightarrow (A \& B), (R \rightarrow S) \rightarrow A, R \rightarrow S, R, S \rightarrow B, (P \rightarrow Q) \rightarrow (S \rightarrow R) \vdash S \leftrightarrow R$ 52

6. $(L \leftrightarrow M) \rightarrow (D \rightarrow A), (A \&(B \& C)) \rightarrow (M \rightarrow L), (C \& A) \& B, (L \rightarrow M) \leftrightarrow (R \leftrightarrow S), S \leftrightarrow R \vdash -D \rightarrow A$ 53
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ESTABLISHING INVALIDITY

As before, we can see that an argument is invalid by noting that any proof of it depends upon the commission of a fallacy. And we still have only one fallacy to watch out for: the Fallacy of Assuming the Consequent.

FAC (Fallacy of Assuming the Consequent): \( \Box \rightarrow O, O \vdash \Box \).
No! Don’t do this! It doesn’t work! This is an invalid inference! Beware!

Practice

Determine whether the arguments are valid or invalid by attempting to prove them. The answers follow in the endnotes.

1. \((P & Q) \rightarrow (R \rightarrow S), P & S, Q \vdash R\) 54
2. \((P & Q) \rightarrow (R \rightarrow S), P & R, Q \vdash S\) 55
3. \((A & B) \rightarrow (L & M), A, B \vdash M\) 56
4. \((A & B) \rightarrow (L & M), L, M \vdash A\) 57
5. “If in order for the mind to be nonphysical, it’s necessary that no computer will ever be able to think, then there is life after death only if no computer will ever be able to think. And in fact, the mind’s being is nonphysical is sufficient for no computer ever being able to think. But computers will always be need to be programmed. Computers will always require human beings to run them. If computers will always need to be programmed and require human beings to run them, then they’ll never be able to think. Therefore there’s life after death.” 58

Let
M = The mind is nonphysical.
N = No computer will ever be able to think.
L = There is life after death.
P = Computers will always need to be programmed.
R = Computers will always require human beings to run them.

6. \((P & Q) \rightarrow (R & S), R \rightarrow (L \rightarrow M), S \rightarrow (M \rightarrow L), P, Q \vdash L \leftrightarrow M\) 59
WHERE NEXT?

So far, we’ve seen the following primitive inference rules:

<table>
<thead>
<tr>
<th>→ O</th>
<th>&amp; O &amp; I</th>
<th>↔ O ↔ I</th>
</tr>
</thead>
</table>

So, what’s missing? If you said → I, give yourself a gold star! That’s the rule that we’ll be getting in the next chapter.

<table>
<thead>
<tr>
<th>→ O → I</th>
<th>&amp; O &amp; I</th>
<th>↔ O ↔ I</th>
</tr>
</thead>
</table>

See you there!
CHAPTER SUMMARY (new material Indicated with “*”)

UNDERSTANDING COMPLEX SENTENCES

* □ and ○ = □ & ○

• □ & ○ is true when and only when both □ and ○ are true.
• If truth “trickles down” over a connector from the whole sentence to the parts of the sentence, the connector is symbolized as “&.”

If □ then ○ = □ → ○

• □ → ○ is false if and only if □ can be true and ○ can be false at the same time.
• Whatever follows “if” (“should” and “provided that”) becomes the antecedent.
• Whatever follows “only if” becomes the consequent.
• Whatever is being described as sufficient becomes the antecedent.
• Whatever is being described as necessary becomes the consequent.

* □ if and only if ○ = □ ↔ ○

• □ ↔ ○ is true when and only when □ and ○ have the same truth value.
• “Necessary and sufficient” = “iff” = “If and only if”

If the sentence has more than one connector, we identify the main connector by
i) Identifying all of the connectors.
ii) Seeing what each connector connects.
iii) If there is nothing left over, that connector may be the main connector. If there is part of the sentence left over, that connector isn’t the main connector.
Once we’ve identified the main connector, we symbolize “around” that main connector

To see if two sentences say the same thing, symbolize the sentences and see if the symbolizations are the same or equivalent to each other (see the equivalence rules below). If so, then the sentences say the same thing. If not, then they don’t.

ASSESSING ARGUMENTS FOR VALIDITY / INVALIDITY

If we can proceed from the premises of an argument to its conclusion by a series of valid inference rules then the argument as a whole is valid. If we can proceed from the premises of an argument to its conclusion only by committing a formal fallacy then the argument as a whole is invalid.

Valid Inference Rules

* & O □ & ○ |- □ & ○ ||- ○
* & I □, ○ |- □ & ○ □, ○ |- ○ & □
* & COM □ & ○ ≡ ○ & □
* & ASSOC □ & (○ & △) ≡ (□ & ○) & △
→ O

*⇔ O

*⇔ I

*⇔ COM

Formal Fallacies

FAC
ANSWERS

Symbolizing with One Connector – Part I

I) Symbolizing with “and”

1. “The earth revolves around the sun and the moon revolves around the earth.”
   Letting “E” represent “the earth revolves around the sun” and “M” represent “the moon revolves around the earth, this would be “E & M.”

2. “The house is large and well-furnished.”
   L & F

3. “He enjoys playing soccer and baseball.”
   S & B

Symbolizing with One Connector – Part II

I) Symbolizing with “and” Equivalents

4. “Dogs are intelligent and affectionate.”
   I & A

5. “Cats are clean, moreover they’re independent.”
   C & I

6. “Guinea pigs are amusing, furthermore they’re social creatures.”
   A & S

7. “Tropical fish are beautiful but they’re difficult to care for.”
   B & D

8. “Squirrels can be a nuisance, yet they’re fun to watch.”
   N & F

9. “Parrots are clever, however they’re messy.”
   C & M

10. “Horses are expensive, nevertheless Ann wants one.”
    E & W

11. “Although hedgehogs are prickly, they’re incredibly cute.”
    P & C
9. “Even though some people are afraid of rats, they make great pets.”
   \[ A \land P \]

10. “Provided that you socialize your puppy well, he should be very friendly.”
    \[ S \rightarrow F \]
    Remember, “provided that” is the same as “if.”

## Symbolizing with More than One Connector

14. “If Hank is a lawyer then he’s both smart and assertive.”
    \[ L \rightarrow (S \land A) \]

15. “Linda is an accountant only if she’s good with numbers, but she always miscalculates the tip.”
    \[ (A \rightarrow G) \land M \]

16. “Being a good listener is a necessary condition for being a good doctor, and a sufficient condition for being a good friend.”
    \[ (D \rightarrow L) \land (L \rightarrow F) \]

17. “Provided that George is both a good programmer and a good communicator, he’ll get the job and he’ll rise quickly through the ranks.”
    \[ (P \land C) \rightarrow (J \land R) \]

## Biconditionals

18. “She’ll pass the class if and only if she studies.”
    \[ P \leftrightarrow S \]

19. “Computers can have minds iff minds are physical.”
    \[ C \leftrightarrow P \]

20. “Completing all of the degree requirements is a necessary and sufficient condition for being eligible to graduate.”
    \[ C \leftrightarrow E \]

21. “Understanding is sufficient for wisdom if and only if understanding is necessary for morality.”
    \[ (U \rightarrow W) \leftrightarrow (M \rightarrow U) \]

22. “The belief that the ethicality of an action is determined by its consequences and the belief that happiness is the ultimate goal of action are jointly a necessary and sufficient condition for the acceptance of utilitarianism.”
    \[ (C \land H) \leftrightarrow U \]
6. “God exists if and only if the universe is a well-ordered place, and the universe is a well-ordered place if and only if it’s amenable to mathematical description.”

\[(G \leftrightarrow W) \& (W \leftrightarrow M)\]

**PRIMATIVE INFERENCE RULES:** &I (and ↔I)

### The &I Rule

1. “Minds causally interact with bodies. Bodies are physical things. And if minds causally interact with bodies and bodies are physical things then minds are physical things. Therefore minds are physical things.

Let \(C\) = “Minds causally interact with bodies,”

\(B\) = “Bodies are physical things,” and

\(M\) = “Minds are physical things.

\[C, B, (C&B) \rightarrow M \vdash M\]

1. \(C\)  \(P\)
2. \(B\)  \(P\)
3. \((C&B) \rightarrow M\)  P - want \(M\)
4. \(C&B\)  1, 2. &I
5. \(M\)  3, 4 →O

2. \(T, S, T \rightarrow M, S \rightarrow L \vdash M\&L\)

1. \(T\)  \(P\)
2. \(S\)  \(P\)
3. \(T \rightarrow M\)  \(P\)
4. \(S \rightarrow L\)  P - want \(M\&L\)
5. \(M\)  1, 3 →O
6. \(L\)  2, 4 →O
7. \(M\&L\)  5, 6 &I

3. \(R \rightarrow S, R \rightarrow T, (S&T) \rightarrow (R \rightarrow L), R \vdash L \& R\)

1. \(R \rightarrow S\)  \(P\)
2. \(R \rightarrow T\)  \(P\)
3. \((S&T) \rightarrow (R \rightarrow L)\)  \(P\)
4. \(R\)  P - want \(L \& R\)
5. \(S\)  1, 4 →O
6. \(T\)  2, 4 →O
7. \(S&T\)  5, 6 &I
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8. R → L 3, 7 → O
9. L 4, 8 → O
10. L & R 4, 9 & I

27 4. [(A → B) & (C → D)] → [(G & F) → E], (A → B) → F, (C → D) → G, A → B, C → D ⊢ E

1. [(A → B) & (C → D)] → [(G & F) → E] P
2. (A → B) → F P
3. (C → D) → G P
4. A → B P
5. C → D P - want E
6. (A → B) & (C → D) 4, 5 & I
7. (G & F) → E 1, 6 → O
8. F 2, 4 → O
9. G 3, 5 → O
10. G & F 8, 9 & I
11. E 7, 10 → O

The ↔ I Rule

28 1. “If utilitarianism is true then an action is good if it maximizes happiness and if utilitarianism is true then an action is good only if it maximizes happiness. Utilitarianism is true. Therefore an action is good if and only if it maximizes happiness.”
Symbolize each sentence as one formula, and let “U” be “utilitarianism is true,” “H” be an action maximizes happiness,” and “G” be an action is morally good.

U → (H → G), U → (G → H), U ⊢ H ↔ G

1. U → (H → G) P
2. U → (G → H) P
3. U P - want H ↔ G
4. H → G 1, 2 → O
5. G → H 2, 3 → O
6. H ↔ G 4, 5 ↔ I

29 2. P → Q, Q → P, (P ↔ Q) → R, (Q → P) → S ⊢ S & R

1. P → Q P
2. Q → P P
3. (P ↔ Q) → R P
4. (Q → P) → S P - want S & R
5. P ↔ Q 1, 2 ↔ I
30. \( A \rightarrow (B \rightarrow C), A, (A \& D) \rightarrow (C \rightarrow B), D \vdash C \leftrightarrow B \)

1. \( A \rightarrow (B \rightarrow C) \)  
2. \( A \)  
3. \( (A \& D) \rightarrow (C \rightarrow B) \)  
4. \( D \)  
5. \( B \rightarrow C \)  
6. \( A \& D \)  
7. \( C \rightarrow B \)  
8. \( C \leftrightarrow B \)  

31. \( H, H \rightarrow M, (H \& M) \rightarrow L, [(H \rightarrow M) \& L] \rightarrow (M \rightarrow H) \vdash M \leftrightarrow H \)

1. \( H \)  
2. \( H \rightarrow M \)  
3. \( (H \& M) \rightarrow L \)  
4. \( [(H \rightarrow M) \& L] \rightarrow (M \rightarrow H) \)  
5. \( M \)  
6. \( H \& M \)  
7. \( L \)  
8. \( (H \rightarrow M) \& L \)  
9. \( M \rightarrow H \)  
10. \( M \leftrightarrow H \)  

32. \( B \rightarrow C, A, A \rightarrow B, (A \& C) \rightarrow (P \rightarrow Q), (B \rightarrow C) \rightarrow (Q \rightarrow P) \vdash P \leftrightarrow Q \)

1. \( B \rightarrow C \)  
2. \( A \)  
3. \( A \rightarrow B \)  
4. \( (A \& C) \rightarrow (P \rightarrow Q) \)  
5. \( (B \rightarrow C) \rightarrow (Q \rightarrow P) \)  
6. \( B \)  
7. \( C \)  
8. \( A \& C \)  
9. \( P \rightarrow Q \)  
10. \( Q \rightarrow P \)  
11. \( P \leftrightarrow Q \)  

33. \( (A \leftrightarrow B) \rightarrow (E \rightarrow C), A \rightarrow B, L, M, (L \& M) \rightarrow (B \rightarrow A), C \rightarrow E, (C \leftrightarrow E) \rightarrow R \vdash R \)
1. \((A \leftrightarrow B) \to (E \to C)\)  
P
2. \(A \to B\)  
P
3. \(L\)  
P
4. \(M\)  
P
5. \((L \& M) \to (B \to A)\)  
P
6. \(C \to E\)  
P
7. \((C \leftrightarrow E) \to R\)  
P - want R
8. \(L \& M\)  
3, 4 &I
9. \(B \to A\)  
5, 8 →O
10. \(A \leftrightarrow B\)  
2, 9 ↔I
11. \(E \to C\)  
1, 10 →O
12. \(C \leftrightarrow E\)  
6, 11 ↔I
13. \(R\)  
7, 12 →O

PRIMATIVE INFERENCE RULES: &O (and ↔ O)

The &O Rule

1. “Scapegoating producing the greatest happiness for the greatest number of people is sufficient for scapegoating being morally correct, and if scapegoating is morally correct then it should be legal. Scapegoating harms a small number of people. Scapegoating makes large numbers of people happy. Scapegoating does produce the greatest happiness for the greatest number of people, if it harms a small number of people while making large numbers of people happy. Therefore, scapegoating should be legal.”

Let

\[ H = \text{Scapegoating produces the greatest happiness for the greatest number of people.} \]
\[ C = \text{Scapegoating is morally correct.} \]
\[ L = \text{Scapegoating should be legal.} \]
\[ S = \text{Scapegoating harms a small number of people.} \]
\[ N = \text{Scapegoating makes large numbers of people happy.} \]

\[(H \to C) \& (C \to L), \ S, N, (S \& N) \to H \vdash L\]

1. \((H \to C) \& (C \to L)\)  
P
2. \(S\)  
P
3. \(N\)  
P
4. \((S \& N) \to H\)  
P - want L
5. \(H \to C\)  
1 &O
6. \(C \to L\)  
1 &O
7. \(S \& N\)  
1, 2 &I
8. \(H\)  
4, 7 →O
9. \(C\)  
5, 8 →O
10. \(L\)  
6, 9 →O
2. S → (Q → T), S & Q |- T & Q

1. S → (Q → T)  P
2. S & Q  P - want T & Q
3. S  2 &O
4. Q → T  1, 3 → O
5. Q  2 &O
6. T  4, 5 → O
7. T & Q  5, 6 &I

3. Q & (P & S), S → W, (W & U) → (R → T), P → U |- (R → T) & (Q & P)

1. Q & (P & S)  P
2. S → W  P
3. (W & U) → (R → T)  P
4. P → U  P - want (R → T) & (Q & P)
5. P & S  1 &O
6. P  5 &O
7. S  5 &O
8. W  2, 7 → O
9. U  4, 6 → O
10. W & U  8, 9 &I
11. R → T  3, 10 → O
12. Q  1 &O
13. Q & P  6, 12 &I
14. (R → T) & (Q & P)  11, 13 &I

4. R → V, [P → (R & S)] & L, L → (Q & P), S → W |- W & V

1. R → V  P
2. [P → (R & S)] & L  P
3. L → (Q & P)  P
4. S → W  P - want W & V
5. L  2 &O
6. P → (R & S)  2 &O
7. Q & P  3, 5 → O
8. P  7 &O
9. R & S  6, 8 → O
10. R  9 &O
11. V  1, 10 → O
12. S  9 &O
13. W  4, 12 → O
14. W & V  11, 13 &I
5. \( L \rightarrow (L \rightarrow (L \rightarrow M)) \), \( (R \rightarrow U) \& (U \rightarrow P) \), \( L \& R \), \( M \rightarrow A \), \( (P \& A) \rightarrow (M \rightarrow D) \) \( \models D \)

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
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<tbody>
<tr>
<td>1</td>
<td>( L \rightarrow (L \rightarrow (L \rightarrow M)) )</td>
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<tr>
<td>2</td>
<td>( (R \rightarrow U) &amp; (U \rightarrow P) )</td>
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<tr>
<td>3</td>
<td>( L &amp; R )</td>
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<tr>
<td>4</td>
<td>( M \rightarrow A )</td>
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<tr>
<td>5</td>
<td>( (P &amp; A) \rightarrow (M \rightarrow D) )</td>
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<tr>
<td>6</td>
<td>( L )</td>
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<tr>
<td>7</td>
<td>( L \rightarrow (L \rightarrow M) )</td>
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<td>8</td>
<td>( L \rightarrow M )</td>
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<td>9</td>
<td>( M )</td>
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<td>10</td>
<td>( R \rightarrow U )</td>
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<td>11</td>
<td>( U \rightarrow P )</td>
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<td>12</td>
<td>( R )</td>
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<td>13</td>
<td>( U )</td>
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<td>14</td>
<td>( P )</td>
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<td>15</td>
<td>( A )</td>
</tr>
<tr>
<td>16</td>
<td>( P &amp; A )</td>
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<tr>
<td>17</td>
<td>( M \rightarrow D )</td>
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<tr>
<td>18</td>
<td>( D )</td>
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</tbody>
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The \( \leftrightarrow \&O \) Rule

39. “Free will is a necessary condition for moral responsibility, and we have free will if and only if not all causes are mechanistic ones. Some of our moral judgments are correct only if we have moral responsibility. In fact some of our moral judgments are correct. Therefore not all causes are mechanistic ones.”

Let
\( R = \text{We have moral responsibility.} \)
\( F = \text{We have free will.} \)
\( N = \text{Not all causes are mechanistic ones.} \)
\( C = \text{Some of our moral judgments are correct.} \)

\((R \rightarrow F) \& (F \leftrightarrow N), C \rightarrow R, C \models N\)

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
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<tbody>
<tr>
<td>1</td>
<td>( (R \rightarrow F) &amp; (F \leftrightarrow N) )</td>
</tr>
<tr>
<td>2</td>
<td>( C \rightarrow R )</td>
</tr>
<tr>
<td>3</td>
<td>( C )</td>
</tr>
<tr>
<td>4</td>
<td>( R )</td>
</tr>
<tr>
<td>5</td>
<td>( R \rightarrow F )</td>
</tr>
<tr>
<td>6</td>
<td>( F )</td>
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<tr>
<td>7</td>
<td>( F \leftrightarrow N )</td>
</tr>
<tr>
<td>8</td>
<td>( F \rightarrow N )</td>
</tr>
</tbody>
</table>
9. \( N \quad 6, 8 \rightarrow O \)

**2. \( (A \& B) \leftrightarrow (C \& D), C, D \vdash A \)**

1. \( (A \& B) \leftrightarrow (C \& D) \quad P \)
2. \( C \quad P \)
3. \( D \quad P - \text{want } A \)
4. \( C \& D \quad 2, 3 \& I \)
5. \( (C \& D) \rightarrow (A \& B) \quad 1 \leftrightarrow O \)
6. \( A \& B \quad 4, 5 \rightarrow O \)
7. \( A \quad 6 \& O \)

**3. \( P \leftrightarrow Q, Q, P \rightarrow (R \leftrightarrow S), R \vdash S \)**

1. \( P \leftrightarrow Q \quad P \)
2. \( Q \quad P \)
3. \( P \rightarrow (R \leftrightarrow S) \quad P \)
4. \( R \quad P - \text{want } S \)
5. \( Q \rightarrow P \quad 1 \leftrightarrow O \)
6. \( P \quad 2, 5 \rightarrow O \)
7. \( R \leftrightarrow S \quad 3, 6 \rightarrow O \)
8. \( R \rightarrow S \quad 7 \leftrightarrow O \)
9. \( S \quad 4, 8 \rightarrow O \)

**DERIVED INFERENCE RULES**

**& COM**

**2. \( (A \& B) \rightarrow (P \& Q), B \& A \vdash Q \& P \)**

1. \( (A \& B) \rightarrow (P \& Q) \quad P \)
2. \( B \& A \quad P - \text{want } Q \& P \)
3. \( A \& B \quad 2 \& \text{COM} \)
4. \( P \& Q \quad 1, 3 \rightarrow O \)
5. \( Q \& P \quad 4 \& \text{COM} \)

**3. \( (A \& B) \rightarrow (P \& Q), B \& A \vdash Q \& P \)**

1. \( (A \& B) \rightarrow (P \& Q) \quad P \)
2. \( B \& A \quad P - \text{want } Q \& P \)
3. \( (B \& A) \rightarrow (P \& Q) \quad 1 \& \text{COM} \)
4. \( (B \& A) \rightarrow (Q \& P) \quad 3 \& \text{COM} \)
5. \( Q \& P \quad 2, 4 \rightarrow O \)
& ASSOC

44. \[(A \& B) \& C \rightarrow [R \& (S \& T)], A \& (B \& C) \vdash (R \& S) \& T\]

1. \([(A \& B) \& C] \rightarrow [R \& (S \& T)] \quad P
2. A \& (B \& C) \quad P \text{ – want } (R \& S) \& T
3. [(A \& B) \& C] \quad 2 \& ASSOC
4. R \& (S \& T) \quad 1, 3 \rightarrow O
5. (R \& S) \& T \quad 4 \& ASSOC

45. \[(A \& B) \& C \rightarrow [R \& (S \& T)], A \& (B \& C) \vdash (R \& S) \& T\]

1. \([(A \& B) \& C] \rightarrow [R \& (S \& T)] \quad P
2. A \& (B \& C) \quad P \text{ – want } (R \& S) \& T
3. [(A \& B) \& C] \rightarrow [(R \& S) \& T] \quad 1 \& ASSOC
4. [A \& (B \& C)] \rightarrow [(R \& S) \& T] \quad 3 \& ASSOC
5. (R \& S) \& T \quad 2, 4 \rightarrow O

& COM and & ASSOC

46. \[(P \& Q) \& R \vdash Q \& (P \& R)\]

1. (P \& Q) \& R \quad P \text{ – want } Q \& (P \& R)
2. (Q \& P) \& R \quad 1 \& COM
3. Q \& (P \& R) \quad 2 \& ASSOC

47. \[(R \& S) \& T, (T \& R) \rightarrow (Q \& (P \& L)) \vdash Q \& L\]

1. (R \& S) \& T \quad P
2. (T \& R) \rightarrow (Q \& (P \& L)) \quad P \text{ – want } Q \& L
3. T \& (R \& S) \quad 1 \& COM
4. (T \& R) \& S \quad 3 \& ASSOC
5. T \& R \quad 4 \& O
6. Q \& (P \& L) \quad 2, 5 \rightarrow O
7. Q \& (L \& P) \quad 6 \& COM
8. (Q \& L) \& P \quad 7 \& ASSOC
9. Q \& L \quad 8 \& O

↔ COM and ↔ ASSOC

48. \[(A \leftrightarrow B) \rightarrow (C \leftrightarrow D), B \leftrightarrow A, \vdash D \rightarrow C\]

1. (A \leftrightarrow B) \rightarrow (C \leftrightarrow D) \quad P
2. B ↔ A   \[\text{P - want D} \leftrightarrow \text{C}\]
3. A ↔ B   \[2 \leftrightarrow \text{COM}\]
4. C ↔ D   \[1, 3 \rightarrow \text{O}\]
5. D ↔ C   \[4 \leftrightarrow \text{COM}\]

\[\text{49} \quad (A \leftrightarrow B) \rightarrow (C \leftrightarrow D), B \leftrightarrow A, \vdash D \rightarrow C\]

1. \[(A \leftrightarrow B) \rightarrow (C \leftrightarrow D)\] \[\text{P}\]
2. B ↔ A   \[\text{P - want D} \leftrightarrow \text{C}\]
3. \[(B \leftrightarrow A) \rightarrow (C \leftrightarrow D)\] \[1 \leftrightarrow \text{COM}\]
4. \[(B \leftrightarrow A) \rightarrow (D \leftrightarrow C)\] \[3 \leftrightarrow \text{COM}\]
5. D ↔ C   \[2, 4 \rightarrow \text{O}\]

\[\text{50} \quad \text{“Francis will be convicted only if Pat tells the truth, provided that there were no witnesses to the crime and Francis left no physical evidence. Francis is smart and the crime occurred on a dark night. If the crime occurred on a dark night and Francis is smart then there were no witnesses to the crime and Francis left no physical evidence. If the police will know that Francis’s alibi is false only if Pat tells the truth then Pat’s telling the truth is sufficient for Francis’s conviction. In fact, for the police to know that Francis’s alibi is false it’s necessary that Pat tell the truth. Therefore, Pat’s telling the truth is necessary and sufficient for Francis’s conviction.”}\]

\[\text{P} = \text{Pat tells the truth.}\]
\[\text{C} = \text{Francis is convicted.}\]
\[\text{N} = \text{There were no witnesses to the crime.}\]
\[\text{L} = \text{Francis left no physical evidence.}\]
\[\text{S} = \text{Francis is smart.}\]
\[\text{D} = \text{The crime occurred on a dark night.}\]
\[\text{F} = \text{The police will know that Francis’s alibi is false}\]

\[(N \& L) \rightarrow (C \rightarrow P), S \& D, (D \& S) \rightarrow (N \& L), (F \rightarrow P) \rightarrow (P \rightarrow C), F \rightarrow P \vdash P \leftrightarrow C\]

1. \[(N \& L) \rightarrow (C \rightarrow P)\] \[\text{P}\]
2. S \& D   \[\text{P}\]
3. \[(D \& S) \rightarrow (N \& L)\] \[\text{P}\]
4. \[(P \rightarrow F) \rightarrow (P \rightarrow C)\] \[\text{P}\]
5. \[P \rightarrow F\] \[\text{P – want P} \leftrightarrow \text{C}\]
6. \[(S \& D) \rightarrow (N \& L)\] \[3 \& \text{Comm}\]
7. N \& L   \[2, 6 \rightarrow \text{O}\]
8. C → P   \[1, 7 \rightarrow \text{O}\]
9. \[P \rightarrow C\] \[4, 5 \rightarrow \text{O}\]
10. \[P \leftrightarrow C\] \[8, 9 \leftrightarrow \text{I}\]
4. “Provided that Alma and Brad were together on Tuesday, Alma is guilty if and only if Brad is. If Alma and Brad were seen leaving the play together then they must have been together on Tuesday. Not only were Alma and Brad seen leaving the play together, but Alma’s fingerprints were on the safe whereas Brad always wears gloves. Alma’s fingerprints being on the safe is sufficient for her guilt and conviction. Should Brad always wear gloves, he’ll go free. Therefore, even though Brad is guilty, he’ll go free.”

T = Alma and Brad were together on Tuesday.
A = Alma is guilty.
B = Brad is guilty.
L = Alma and Brad were seen leaving the play together.
D = Alma’s fingerprints were on the safe.
G = Brad always wears gloves.
C = Alma will be convicted.
F = Brad will go free.

\( T \rightarrow (A \leftrightarrow B), L \rightarrow T, L \& (D \& G), D \rightarrow (A \& C), G \rightarrow F \mid B \& F \)

1. \( T \rightarrow (A \leftrightarrow B) \) 
2. \( L \rightarrow T \),  
3. \( L \& (D \& G) \) 
4. \( D \rightarrow (A \& C) \) 
5. \( G \rightarrow F \) 
6. \( L \) 
7. \( D \& G \) 
8. \( D \) 
9. \( G \) 
10. \( T \) 
11. \( A \leftrightarrow B \) 
12. \( A \& C \) 
13. \( A \) 
14. \( A \rightarrow B \) 
15. \( B \) 
16. \( F \) 
17. \( B \& F \) 

5. \( (P \leftrightarrow Q) \leftrightarrow (A \& B), (R \rightarrow S) \rightarrow A, R \rightarrow S, R \rightarrow B, (P \rightarrow Q) \rightarrow (S \rightarrow R) \mid S \leftrightarrow R \)

1. \( (P \leftrightarrow Q) \leftrightarrow (A \& B) \) 
2. \( (R \rightarrow S) \rightarrow A \) 
3. \( R \rightarrow S \) 
4. \( R \) 
5. \( S \rightarrow B \) 
6. \( (P \rightarrow Q) \rightarrow (S \rightarrow R) \) 

- want \( S \leftrightarrow R \)
Chapter 2 - Conjunctions and Biconditionals

Dona Warren, Department of Philosophy, The University of Wisconsin – Stevens Point

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<tbody>
<tr>
<td>7. A</td>
<td>2, 3 →O</td>
<td></td>
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<tr>
<td>8. S</td>
<td>3, 4 →O</td>
<td></td>
</tr>
<tr>
<td>9. B</td>
<td>5, 8 →O</td>
<td></td>
</tr>
<tr>
<td>10. A &amp; B</td>
<td>7, 9 &amp;I</td>
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<tr>
<td>11. (A &amp; B) → (P ↔ Q)</td>
<td>1 ↔O</td>
<td></td>
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<tr>
<td>12. P ↔ Q</td>
<td>10, 11 →O</td>
<td></td>
</tr>
<tr>
<td>13. P → Q</td>
<td>12 ↔O</td>
<td></td>
</tr>
<tr>
<td>14. S → R</td>
<td>6, 13 →O</td>
<td></td>
</tr>
<tr>
<td>15. S ↔ R</td>
<td>3, 14 ↔I</td>
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\[ (L \leftrightarrow M) \rightarrow (D \rightarrow A), (A \& (B \& C)) \rightarrow (M \rightarrow L), (C \& A) \& B, (L \rightarrow M) \leftrightarrow (R \leftrightarrow S), S \leftrightarrow R \vdash D \rightarrow A \]

1. (L \leftrightarrow M) \rightarrow (D \rightarrow A) | P
2. (A \& (B \& C)) \rightarrow (M \rightarrow L) | P
3. (C \& A) \& B | P
4. (L \rightarrow M) \leftrightarrow (R \leftrightarrow S) | P
5. S \leftrightarrow R | P want D \rightarrow A
6. C \& (A \& B) | 3 \& Assoc
7. (A \& B) \& C | 6 \& Comm
8. A \& (B \& C) | 7 \& Assoc
9. M \rightarrow L | 2, 8 \rightarrow O
10. R \leftrightarrow S | 5 \leftrightarrow Comm
11. (R \leftrightarrow S) \rightarrow (L \rightarrow M) | 4 \leftrightarrow O
12. L \rightarrow M | 10, 11 \rightarrow O
13. L \leftrightarrow M | 9, 12 \leftrightarrow I
14. D \rightarrow A | 1, 13 \rightarrow O

**ESTABLISHING INVALIDITY**

\[ (P \& Q) \rightarrow (R \rightarrow S), \ P \& S, \ Q \vdash R \]

1. (P \& Q) \rightarrow (R \rightarrow S) | P
2. P \& S | P
3. Q | P – want R
4. P | 2 \& O
5. P \& Q | 3, 4 \& I
6. R \rightarrow S | 1, 5 \rightarrow O
7. S | 2 \& O
8. R | 6, 7 FAC! No

Invalid

\[ (P \& Q) \rightarrow (R \rightarrow S), \ P \& R, \ Q \vdash S \]

54 1. (P & Q) → (R → S), P & S, Q ⊢ R

55 2. (P & Q) → (R → S), P & R, Q ⊢ S
1. \((P \land Q) \rightarrow (R \rightarrow S)\)
2. \(P \land R\)
3. \(Q\)
4. \(P\)
5. \(P \land Q\)
6. \(R \rightarrow S\)
7. \(R\)
8. \(S\)

Valid

56. \((A \land B) \rightarrow (L \land M), A, B \vdash M\)

1. \((A \land B) \rightarrow (L \land M)\)
2. \(A\)
3. \(B\)
4. \(A \land B\)
5. \(L \land M\)
6. \(M\)

Valid

57. \((A \land B) \rightarrow (L \land M), L, M \vdash A\)

1. \((A \land B) \rightarrow (L \land M)\)
2. \(L\)
3. \(M\)
4. \(L \land M\)
5. \(A \land B\)
6. \(A\)

Invalid

58. “If in order for the mind to be nonphysical, it’s necessary that no computer will ever be able to think, then there is life after death only if no computer will ever be able to think. And in fact, the mind’s being is nonphysical is sufficient for no computer ever being able to think. But computers will always need to be programmed. Computers will always require human beings to run them. If computers will always need to be programmed and require human beings to run them, then they’ll never be able to think. Therefore there’s life after death.”

Let
\(M = \text{The mind ins nonphysical.}\)
\(N = \text{No computer will ever be able to think.}\)
\(L = \text{There is life after death.}\)
\(P = \text{Computers will always need to be programmed.}\)
R = Computers will always require human beings to run them.

\[(M \rightarrow N) \rightarrow (L \rightarrow N), M \rightarrow N, P, R, (P&R) \rightarrow N \vdash L\]

1. \( (M \rightarrow N) \rightarrow (L \rightarrow N) \) \( P \)
2. \( M \rightarrow N \) \( P \)
3. \( P \) \( P \)
4. \( R \) \( P \)
5. \( (P&R) \rightarrow N \) \( P \) – want \( L \)
6. \( P \& R \) \( 3, 4 \ \& I \)
7. \( N \) \( 5, 6 \rightarrow O \)
8. \( L \rightarrow N \) \( 1, 2 \rightarrow O \)
9. \( L \) \( 7, 8 \ \text{FAC! No.} \)

Invalid

\[59 \] 6. \( (P \& Q) \rightarrow (R \& S), R \rightarrow (L \rightarrow M), S \rightarrow (M \rightarrow L), P, Q \vdash L \leftrightarrow M \]

1. \( (P \& Q) \rightarrow (R \& S) \) \( P \)
2. \( R \rightarrow (L \rightarrow M) \) \( P \)
3. \( S \rightarrow (M \rightarrow L) \) \( P \)
4. \( P \) \( P \)
5. \( Q \) \( P \) – want \( L \leftrightarrow M \)
6. \( P \& Q \) \( 4, 5 \ \& I \)
7. \( R \& S \) \( 1, 6 \rightarrow O \)
8. \( R \) \( 7 \ \& O \)
9. \( S \) \( 8 \ \& O \)
10. \( L \rightarrow M \) \( 2, 8 \rightarrow O \)
11. \( M \rightarrow L \) \( 3, 9 \rightarrow O \)
12. \( L \leftrightarrow M \) \( 10, 11 \leftrightarrow I \)

Valid