Conditional Propositions and Logical Equivalence

Section 1.2

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Math 209 - Fall 2008

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1 Conditional Propositions

1.1 Conditional Propositions

Conditional Propositions

- “If it rains this afternoon, then I will carry an umbrella” is a proposition

- Is the proposition true or false?
  - *True* if it rains and I carry an umbrella.
  - *False* if it rains and I don’t carry an umbrella.
  - What if it doesn’t rain?

- If \( p \) and \( q \) are propositions, the proposition “if \( p \) then \( q \)” is a conditional proposition.
  - Denoted \( p \rightarrow q \)
  - \( p \) is the hypothesis or antecedent.
  - \( p \) is also called a sufficient condition.
  - \( q \) is the conclusion or consequent.
  - \( q \) is also called a necessary condition.
  - \( p \rightarrow q \) is another binary operator.

1.2 Truth Table of Conditional Propositions

Truth Table of \( p \rightarrow q \)

- When is \( p \rightarrow q \) true?
  - *True* if both \( p \) and \( q \) are true.
  - *False* if \( p \) is true, but \( q \) is false.
  - *True* if \( p \) is false.
  - Referred to as either:
    * True by default,
    * Vacuously true,
* Trivially true

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p → q</th>
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</thead>
<tbody>
<tr>
<td>T</td>
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Truth Table of \( p \rightarrow q \)

Example. “If Brett Favre is the starting quarterback for the Packers, then \( 2 + 2 = 5 \)” is a true proposition.

Problem. Complete the truth table.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>( \neg(p \rightarrow q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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1.3 Conditional Propositions in Computing

The “if-then” Statement

- There is no direct analog to \( p \rightarrow q \) in Java.
- Java does have an “if-then” statement

```java
if (condition){
    statement
}
```

- If \( condition \) is true, then \( statement \) executes
- If \( condition \) is false, then \( statement \) is irrelevant.
1.4 The Converse

The Converse

Problem. Complete the truth table.

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<tr>
<th>$p$</th>
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<th>$q \to p$</th>
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- Let $p \to q$ be a conditional proposition. The converse of $p \to q$ is $q \to p$
- The converse is not the same as the original proposition.

The Converse

Example. 
- “If Brett Favre is the starting quarterback for the Packers, then $2 + 2 = 4$” is a true proposition.
- “If $2 + 2 = 4$, then Brett Favre is the starting quarterback for the Packers” is a false proposition.
- These propositions are converses of each other.

1.5 Biconditional Proposition

Biconditional Proposition

- If $p$ and $q$ are propositions, then “$p$ if and only if $q$” or “$p$ iff $q$” is a biconditional proposition.
- We denote it by $p \leftrightarrow q$
- $p \leftrightarrow q$ is true precisely when $p$ and $q$ have the same truth values

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2 Logical Equivalence

2.1 Logical Equivalence

Logically Equivalent Propositions

- If $P$ and $Q$ are compound propositions built from $p_1, p_2, \ldots, p_n$, then $P$ and $Q$ are logically equivalent provided that $P$ and $Q$ have the same truth values, no matter what truth values $p_1, p_2, \ldots, p_n$ have.
  - We denote this $P \equiv Q$
  - This is the same as $P \leftrightarrow Q$ being a tautology.

Example.
- $\neg (p \rightarrow q) \equiv p \land \neg q$
- $p \rightarrow q \equiv q \rightarrow p$

Logically Equivalent Propositions

Example. $p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$

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2.2 De Morgan’s Laws

De Morgan’s Laws

Theorem 1. De Morgan’s Laws

- $\neg (p \lor q) \equiv \neg p \land \neg q$
- $\neg (p \land q) \equiv \neg p \lor \neg q$

- Augustus De Morgan was a 19th century British mathematician, born in India.
- Gives a way of negating conjunctions and disjunctions.
2.3 The Contrapositive

The Contrapositive

- The contrapositive of a proposition \( p \rightarrow q \) is the proposition \( \neg q \rightarrow \neg p \).

**Theorem 2.** The conditional proposition \( p \rightarrow q \) is logically equivalent to its contrapositive \( \neg q \rightarrow \neg p \).

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<th>( p )</th>
<th>( q )</th>
<th>( \neg(p \lor q) )</th>
<th>( \neg p \land \neg q )</th>
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**Summary**

You should be able to:

- Use *conditional* and *biconditional* propositions.
- Use the *converse* and *contrapositive* of a statement.
- Identify *logically equivalent* propositions.
- Use *De Morgan's Laws*.