An almost sure limit theorem for the product of partial sums with stable distribution

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ABSTRACT. We prove an almost sure limit theorem for the product of partial sums of random variables in the domain of attraction of an α-stable law when $1 < \alpha < 2$.

1. Introduction and Main Result

Arnold and Villaseñor (1998) proved the following central limit theorem for the partial sum S_n of a sequence of exponential random variables of $(X_n)_{n\geq 1}$ with mean 1:

$$\frac{1}{\sqrt{2n}}\left(\sum_{k=1}^{n}\log(S_k) - n\log(n) + n\right) \xrightarrow{\mathcal{D}} \mathcal{N} \text{ as } n\to\infty$$

where $\mathcal{N}$ is a standard normal variable.

Rempała and Wesołowski (2002) proved that (1.1) was still valid for any sequence of square integrable, independent and identically distributed (i.i.d.) positive random variables. Later Qi (2003) considered a sequence of random variables $(X_n)_{n\geq 1}$ in the domain of attraction of a stable law $\mathcal{L}$ with index α when $1 < \alpha \leq 2$ and proved that there existed a numerical sequence $(a_n)_{n\geq 1}$ such that

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{\mu/a_n} \xrightarrow{\mathcal{D}} e^{\Gamma(\alpha+1)^{1/\alpha}\mathcal{L}} \text{ as } n\to\infty \tag{1.1}$$

where $\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x}dx$. A particular case of (1.1) when $\alpha = 2$ was proved by Rempała and Wesołowski (2002).

The goal of this paper is to obtain an almost sure version of (1.1) when $1 < \alpha < 2$. The case when $\alpha = 2$ was proved by Gonchigdanzan and Rempała (2006). Our main result is the following almost sure limit theorem:

Theorem 1.1. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. positive random variables in the domain of attraction of an α-stable law of $\mathcal{L}$ with $1 < \alpha < 2$ and $\mathbf{E}(X_1) = \mu$. Then there exists a sequence of positive real numbers $(a_n)_{n\geq 1}$ such that for any real x*

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbf{I}\left(\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{\mu/a_n} \leq x\right) \xrightarrow{a.s.} F(x) \text{ as } N\to\infty. \tag{1.2}$$

where F is the distribution function of the random variable $e^{\Gamma(\alpha+1)^{1/\alpha}\mathcal{L}}$.

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It is known that classic central limit theorem does not imply almost sure limit theorem in general. We refer to the survey paper Berkes (1998) for more detail discussions.

2. Auxiliary Results and Proofs

2.1. Existence of (a_n) in Theorem 1.1.

It is well known from the theory of stable distribution that there exist a sequence of positive real numbers (a_n) and a sequence of (b_n) that satisfy

$$\frac{S_n - b_n}{a_n} \xrightarrow{\mathcal{D}} \mathcal{L} \text{ as } n \to \infty \tag{2.1}$$

where $\mathcal{L}$ is an α-stable distribution for $1 < \alpha < 2$.

In fact, we show that (1.2) holds for any sequence of (a_n) that satisfies the above weak convergence. □

2.2. Auxiliary Results I: Independent Random Variables

First, let us prove two theorems for i.i.d. random variables which are not necessarily in a stable law.

Throughout the paper $a \ll b$ stands for $a = \mathrm{O}(b)$.

Theorem 2.1. *Let $Y_1, Y_2, \ldots$ be independent random variables and let $(d_n)_{n\geq 1}$ be a positive numerical sequence such that*

$$E\left|\frac{1}{d_n}\sum_{k=1}^{n} b_{k,n}(Y_k - \mu)\right| = \mathrm{O}(\log n) \tag{2.2}$$

where $b_{k,n} = \sum_{k\leq j\leq n} 1/j$. Assume that

$$\frac{d_l}{d_k} \gg \left(\frac{l}{k}\right)^{\gamma} \quad (l \geq k \geq n_0) \tag{2.3}$$

for some $\gamma > 0$ and $n_0 \geq 1$. Then for any distribution G,

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbf{I}\left(\frac{1}{d_n}\sum_{k=1}^{n} b_{k,n}(Y_k - \mu) \leq x\right) \xrightarrow{a.s.} G(x) \quad \text{as} \quad N \to \infty \tag{2.4}$$

if and only if

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbf{P}\left(\frac{1}{d_n}\sum_{k=1}^{n} b_{k,n}(Y_k - \mu) \leq x\right) \longrightarrow G(x) \quad \text{as} \quad N \to \infty. \tag{2.5}$$

where $\mathbf{I}$ denotes the indicator function.

Proof. Denote that $S_{n,n} = \sum_{1\leq k\leq n} b_{k,n}(Y_k - \mu)$ and $\xi_k = f(S_{k,k}/d_k) - \mathbf{E}f(S_{k,k}/d_k)$. It is known that $S_{n,n}/d_n \xrightarrow{\mathcal{D}} X$ for a random variable X if and only if $\mathbf{E}f(S_{n,n}/d_n) \to \mathbf{E}f(X)$ for any bounded Lipschitz function f as $n \to \infty$. Hence to prove the theorem it suffices to show that

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\xi_k \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty. \tag{2.6}$$

However the following is sufficient for (2.6) (Berkes and Dehling (1993), p. 1648):

$$\mathbf{E}\Bigl(\sum_{k=1}^{n}\frac{1}{k}\xi_k\Bigr)^2 = \mathrm{O}(\log^{2-\delta} n) \ \text{ for some } \ \delta > 0. \tag{2.7}$$

Therefore, we will show (2.7) to prove the theorem.

Writing $S_{l,l}-S_{k,k} = b_{k+1,l}S_k+\bigl[b_{k+1,l}(Y_{k+1}-\mu)+\cdots+b_{ll}(Y_l-\mu)\bigr]$ for $l \geq k$, we see that $S_{l,l}-S_{k,k}-b_{k+1,l}S_k$ is independent of $S_{k,k}$ and hence

$$\mathbf{C}ov\biggl(f\Bigl(\frac{S_{k,k}}{d_k}\Bigr), f\Bigl(\frac{S_{l,l}-S_{k,k}-b_{k+1,l}S_k}{d_l}\Bigr)\biggr) = 0 \ \text{ for } \ l \geq k.$$

Thus for $l \geq k$

$$\bigl|\mathbf{E}(\xi_k\xi_l)\bigr| \ll \biggl|\mathbf{C}ov\biggl(f\Bigl(\frac{S_{k,k}}{d_k}\Bigr), f\Bigl(\frac{S_{l,l}}{d_l}\Bigr) - f\Bigl(\frac{S_{l,l}-S_{k,k}-b_{k+1,l}S_k}{d_l}\Bigr)\biggr|.$$

By the Lipschitz property of f, (2.2), and (2.3) we have

$$\begin{aligned}\bigl|\mathbf{E}(\xi_k\xi_l)\bigr| &\ll \mathbf{E}\left(\frac{|S_{k,k}+b_{k+1,l}S_k|}{d_l}\right) \ll \mathbf{E}\left(\frac{|S_{k,k}|}{d_l}\right) + \mathbf{E}\left(\frac{|b_{k+1,l}S_k|}{d_l}\right) \ll \\ &\ll \frac{d_k}{d_l}\mathbf{E}\left(\frac{|S_{k,k}|}{d_k}\right) + b_{k+1,l}\frac{d_k}{d_l}\mathbf{E}\left(\frac{|S_k|}{d_k}\right) \ll \\ &\ll \frac{d_k}{d_l}\log k + b_{k+1,l}\frac{d_k}{d_l} \ll \left(\frac{k}{l}\right)^{\gamma}\log k + b_{k+1,l}\left(\frac{k}{l}\right)^{\gamma}.\end{aligned}$$

Note that $b_{k+1,l} = \mathrm{O}(\log(l/k))$. By choosing $\epsilon > 0$ so that $b_{k+1,l} < (l/k)^{\epsilon}$ and $\epsilon < \gamma$ we have the following estimate

$$\bigl|\mathbf{E}(\xi_k\xi_l)\bigr| \ll \left(\frac{k}{l}\right)^{\gamma}\log k + \left(\frac{k}{l}\right)^{\gamma'} \ll \left(\frac{k}{l}\right)^{\gamma'}\log k$$

where $\gamma' = \gamma - \epsilon$.

On the other hand we have a trivial estimate of $\mathbf{E}(\xi_k\xi_l) \leq K^2$ because ξ_k is bounded. These two estimates of $\mathbf{E}(\xi_k\xi_l)$ will be used as follows:

$$\mathbf{E}(\xi_k\xi_l) \ll \begin{cases} 1, & \text{if } \ l/k \leq \exp\bigl((\log n)^{1-\delta}\bigr) \\ (k/l)^{\gamma'}\log k, & \text{if } \ l/k \geq \exp\bigl((\log n)^{1-\delta}\bigr) \end{cases}$$

where δ is any positive number and $\exp(t) = e^t$.

Using the above estimates

$$\sum_{\substack{1\leq l\leq k\leq n \\ l/k\leq\exp{(\log n)^{1-\delta}}}} \frac{\mathbf{E}(\xi_k\xi_l)}{kl} \leq \sum_{1\leq k\leq n}\frac{1}{k}\sum_{k\leq l\leq ke^{(\log n)^{1-\delta}}}\frac{1}{l} \ll \sum_{k=1}^{n}\frac{1}{k}\log^{1-\delta} n \ll \log^{2-\delta} n \tag{2.8}$$

and

$$\sum_{\substack{1\leq l\leq k\leq n \\ l/k\geq\exp{(\log n)^{1-\delta}}}} \frac{\mathbf{E}(\xi_k\xi_l)}{kl} \leq e^{-\gamma'(\log n)^{1-\delta}}\log n \sum_{1\leq k\leq l\leq n}\frac{1}{kl} \leq e^{-\gamma'(\log n)^{1-\delta}}\log^3 n \ll \log^{2-\delta} n. \tag{2.9}$$

Thus (2.8) and (2.9) together give (2.7). $\square$

Theorem 2.2. *Let* $(Y_n)_{n\geq 1}$ *be a sequence of i.i.d. random variables. Denote* $S_n = \sum_{j=1}^n Y_j$. *Assume that* $E|S_n| < \infty$ *for all* $n \geq 1$. *Then, for any real numbers of* $\lambda_1, \lambda_2, \dots, \lambda_n$ *assuming* $\lambda_0 = 0$ *the following inequality holds:*

$$E\Big|\sum_{j=1}^n \lambda_j\, Y_j\Big| \leq E|S_n| \sum_{j=1}^n |\lambda_j - \lambda_{j-1}|.$$

Proof. Observe that

$$\sum_{j=1}^n \lambda_j Y_j = \sum_{j=1}^n (Y_j + Y_{j+1} + \dots + Y_n)(\lambda_j - \lambda_{j-1}).$$

Note that $E|S_m| \leq E|S_n|$ for $m \leq n$ because $|S_n|$ is a submartingale. Since $(Y_j)_{j\geq 1}$ are i.i.d.,

$$E\Big|\sum_{j=1}^n \lambda_j Y_j\Big| = \sum_{j=1}^n E|Y_j + Y_{j+1} + \dots + Y_n||\lambda_j - \lambda_{j-1}| \leq E|S_n| \sum_{j=1}^n |\lambda_j - \lambda_{j-1}|. \quad \square$$

2.2. Auxiliary Results II: Random Variables with Stable Distribution

The following settings and assumptions will used throughout the rest of the paper:

a. $(X_n)_{n\geq 1}$ is a sequence of i.i.d. positive random variables in the domain of attraction of an α-stable law of $\mathcal{L}$ with $1 < \alpha < 2$ with $\mathbf{E}(X_1) = \mu$

b. $S_n = X_1 + X_2 + \dots + X_n$ and $S_{n,n} = \sum_{1\leq k\leq n} b_{k,n}(X_k - \mu)$

c. $(a_n)_{n\geq 1}$ is a sequence that satisfies (2.1)

d. $b_{k,n} = \sum_{k\leq j\leq n} 1/j$.

Now we are going to prove three lemmas that will be needed for the proof of our main theorem.

Lemma 2.1. *We have*

$$E\Big|\frac{1}{a_n}\sum_{j=1}^n \log\left(\frac{n+1}{j}\right)(X_j - \mu)\Big| = \mathrm{O}(\log n).$$

Proof. By Theorem 6.1 in DeAscota and Giné (1979) we have

$$E\Big|\frac{1}{a_n}\sum_{j=1}^n (X_j - \mu)\Big| = \mathrm{O}(1).$$

Hence, by applying Theorem 2.2 when $\lambda_j = \log j$,

$$E\Big|\frac{1}{a_n}\sum_{j=1}^n \log\Big(\frac{n+1}{j}\Big)(X_j - \mu)\Big| \leq E\Big|\frac{1}{a_n}\sum_{j=1}^n \log(n+1)(X_j - \mu)\Big| + E\Big|\frac{1}{a_n}\sum_{j=1}^n \log j(X_j - \mu)\Big| \ll$$
$$\ll E\Big|\frac{1}{a_n}\sum_{j=1}^n (X_j - \mu)\Big|\log(n+1) + E\Big|\frac{1}{a_n}\sum_{j=1}^n (X_j - \mu)\Big|\sum_{j=1}^{n-1}\Big|\log(j+1) - \log j\Big| \ll \log n. \quad \square$$

Lemma 2.2. *We have*

$$\frac{1}{\log N}\sum_{n=1}^N \frac{1}{n}\mathbf{I}\left(\frac{1}{a_n}S_{n,n} \leq x\right) \xrightarrow{a.s.} \Gamma(\alpha+1)^{1/\alpha}\mathcal{L} \quad \text{as} \quad N \to \infty. \tag{2.10}$$

Proof. By Lemma 2.3 in Qi (2003) we have

$$\frac{1}{a_n}\sum_{k=1}^{n}\log\left(\frac{n+1}{k}\right)(X_k-\mu)\xrightarrow{\mathcal{D}}\Gamma(\alpha+1)^{1/\alpha}\mathcal{L}\quad\text{as}\quad n\to\infty.$$

Hence using the fact that $\log\left(\frac{n+1}{k}\right)=\mathrm{O}(b_{k,n})$ we have

$$\frac{1}{a_n}S_{n,n}=\frac{1}{a_n}\sum_{k=1}^{n}\frac{b_{k,n}}{\log\left(\frac{n+1}{k}\right)}\log\left(\frac{n+1}{k}\right)(X_k-\mu)\xrightarrow{\mathcal{D}}\Gamma(\alpha+1)^{1/\alpha}\mathcal{L}$$

that implies

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbf{P}\left(\frac{S_{n,n}}{a_n}\le x\right)\longrightarrow\Gamma(\alpha+1)^{1/\alpha}\mathcal{L}\quad\text{as}\quad N\to\infty. \tag{2.11}$$

By Lemma 2.1 the condition (2.2) of Theorem 2.1 holds. It is known that a_n satisfies (2.3) (Berkes and Dehling (1993), p. 1644). Thus by Theorem 2.1, (2.11) is equivalent to (2.10). □

Lemma 2.3. *We have*

$$\left|\frac{1}{a_n}\sum_{k=1}^{n}\log\left(\frac{S_k}{\mu k}\right)-\frac{1}{a_n}\sum_{k=1}^{n}\left(\frac{S_k}{\mu k}-1\right)\right|\xrightarrow{a.s.}0\quad\text{as}\quad n\to\infty.$$

Proof. We use the following facts to prove the lemma:

a. *Marcinkiewicz-Zygmund* SLLN: $(S_n-n\mu)\overset{a.s.}{=}\mathrm{o}(n^{1/p})$ when $\mathbf{E}X_1^p<\infty$ for $1\le p<2$ (Chow and Teicher (1997), p. 125).

b. $\log(1+x)-x=\mathrm{O}(x^2)$ for $|x|<1/2$

c. a_n can always be given in the form of $a_n=n^{1/\alpha}L(n)$ where L is a slowly varying function. Hence, $n^\beta=\mathrm{o}(a_n)$ when $\beta<1/\alpha$ and also $n^{2/p-1}/a_n\xrightarrow{a.s.}0$ for any $p\in\left(\frac{2\alpha}{1+\alpha},\alpha\right)$ (Qi (2003), p. 98).

Since $S_k/(\mu k)-1\xrightarrow{a.s.}0$, the above facts together give

$$\left|\frac{1}{a_n}\sum_{k=1}^{n}\log\left(\frac{S_k}{\mu k}\right)-\frac{1}{a_n}\sum_{k=1}^{n}\left(\frac{S_k}{\mu k}-1\right)\right|\ll\frac{1}{a_n}\sum_{k=1}^{n}\left(\frac{S_k-k\mu}{k\mu}\right)^2\ll\frac{n^{2/p-1}}{a_n}\xrightarrow{a.s.}0.\quad\square$$

2.4. Proof of Main Result

We are now ready to prove our main result Theorem 1.1. The main idea of the proof is similar to the proof of the ASCLT in Gonchigdanzan and Rempała (2006). We can write

$$\frac{1}{a_n}S_{n,n}=\frac{1}{a_n}\sum_{k=1}^{n}b_{k,n}(X_k-\mu)=\frac{\mu}{a_n}\sum_{k=1}^{n}\left(\frac{S_k}{\mu k}-1\right).$$

By Lemma 2.2 and Lemma 2.3 we have

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbf{I}\left(\frac{\mu}{a_n}\sum_{k=1}^{n}\left(\frac{S_k}{\mu k}-1\right)\le x\right)\xrightarrow{a.s.}\Gamma(\alpha+1)^{1/\alpha}\mathcal{L}\ \text{as}\ N\to\infty$$

and

$$\mathbf{I}\left(\frac{\mu}{a_n}\sum_{k=1}^{n}\log\left(\frac{S_k}{\mu k}\right)\le x\right)-\mathbf{I}\left(\frac{\mu}{a_n}\sum_{k=1}^{n}\left(\frac{S_k}{\mu k}-1\right)\le x\right)\xrightarrow{a.s.}0 \text{ as } n\to\infty.$$

Hence we have

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbf{I}\left(\frac{\mu}{a_n}\sum_{k=1}^{n}\log\left(\frac{S_k}{\mu k}\right)\le x\right)\xrightarrow{a.s.}\Gamma(\alpha+1)^{1/\alpha}\mathcal{L} \text{ as } N\to\infty.$$

Thus, noting

$$\frac{\mu}{a_n}\sum_{k=1}^{n}\log\left(\frac{S_k}{\mu k}\right)=\log\left(\frac{\prod_{k=1}^{n}S_k}{n!\mu^n}\right)^{\mu/a_n}$$

we have

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbf{I}\left(\log\left(\frac{\prod_{k=1}^{n}S_k}{n!\mu^n}\right)^{\mu/a_n}\le x\right)\xrightarrow{a.s.}\Gamma(\alpha+1)^{1/\alpha}\mathcal{L} \text{ as } N\to\infty$$

that is equivalent to

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbf{I}\left(\left(\frac{\prod_{k=1}^{n}S_k}{n!\mu^n}\right)^{\mu/a_n}\le x\right)\xrightarrow{a.s.}e^{\Gamma(\alpha+1)^{1/\alpha}\mathcal{L}} \text{ as } N\to\infty. \quad \square$$

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