CONSTRUCTING ALGEBRAIC LINKS FOR LOW EDGE NUMBERS

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ABSTRACT

A method is given for economically constructing any algebraic knot or link K. This construction, which involves tree diagrams, gives a new upper bound for the edge number of K that is proven to be at most twice the crossing number of K. Furthermore, it realizes a minimal-crossing projection.

Keywords: knot, link, edge number, stick number, algebraic link, arborescent link

1. Introduction

The edge number of a piecewise-linear link K, denoted $e(K)$, is the minimum number of edges or linear segments required to form a link of type K where the edges may be of any length. This link invariant has also been called stick number and polygonal index. The natural goal is to discern the edge number of every link type. To approach that goal, upper and lower bounds on edge number are sought for more and more classes of links. These bounds often compare edge number to crossing number, $c(K)$.

The most general upper bound can be found in Negami [1]. It is that $e(K) \leq 2c(K)$ for any nontrivial link K other than the Hopf link, which has edge number 6 and crossing number 2. Sharper upper bounds have been found for various classes of links (such as torus knots and their compositions) by Adams et al. [2], Jin [3], Furstenberg et al. [4], McCabe [5], and Rawdon and Scharein [6]. Also, precise edge numbers are known for a collection of knots with small crossing number due to the work of Randell [7], Meissen [8], Millett [9], Scharein [10], and Calvo [11].

In this paper, constructive methods are used to find a new upper bound on edge numbers for the class of algebraic links. A combination of the descriptions of this class given by Conway in [12] and by Gabai in [13] are used. As Conway writes, any link projection can be viewed as a collection of algebraic tangles, and a link which possesses a projection consisting of only one algebraic tangle is called an *algebraic link*. Conway also defines the graph that shows the way in which the algebraic tangles of an arbitrary link are connected to be a *basic polyhedron*. Once piecewise-linear configurations of basic polyhedra are found, the results of this paper can be carried further and generalized to all links.

Section 2 of this paper describes algebraic links in more detail and gives ways to represent them. Section 3 gives a procedure for putting a (smooth) projection of an algebraic tangle into a standard form. In Section 4 we construct a piecewise-linear (PL) version of algebraic links while giving a bound on their edge numbers. This new bound is compared with Negami's known bound of $e(K) \leq 2c(K)$ in Section 5, and then examples of the bounds achieved are shown for a sample of specific knots of low crossing number.

2. Descriptions of Algebraic Links

Given a minimal-crossing projection of a knot or link K, we can identify series of crossings (turned until they are horizontal or vertical) called integral tangles. We can also see how these integral tangles are connected to one another. Each tangle has four arcs emanating from it. If two arcs from one tangle connect to two arcs from another, then the two tangles are connected algebraically.

An integral tangle is the most basic type of algebraic tangle, and if two integral tangles are connected as just described, then together they form an algebraic tangle. Also, if any two algebraic tangles are connected in this way, then a new algebraic tangle is formed. Figure 1 shows an integral tangle, an algebraic tangle made from two integral tangles, and a more complicated algebraic tangle, which is made up of smaller algebraic tangles that are themselves connected algebraically.

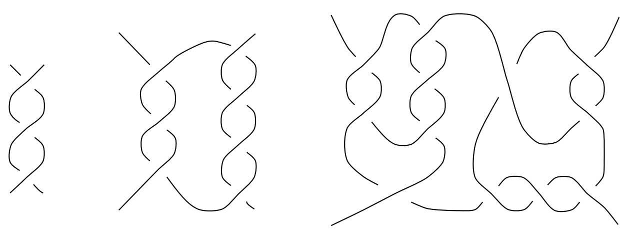

Fig. 1. Three examples of algebraic tangles

If the ends of an algebraic tangle are closed off with two arcs, then an algebraic knot or link is formed. An *algebraic link* is one which has a projection that is simply a closed-off algebraic tangle. The description just given is similar to the one given by Conway in [12] but without his notation.

A non-algebraic projection of a link would be one which consisted of at least six algebraic tangles, no two of which were connected to one another by more than one arc. To see the connections more clearly, we could shrink each algebraic tangle in a given projection to a point. If two of these vertices are connected to each other by two arcs, then their tangles were part of a larger algebraic tangle, so they should be shrunk together into a single point. Once this type of simplifying is done, you have the basic polyhedron [12] for that projection.

Notice that a link is algebraic if it has at least one projection which is algebraic, but this does not mean that all its projections will be algebraic. For example, the minimal-crossing projection of knot 8_{19} found in Rolfsen's table [14] is not algebraic since its basic polyhedron has six vertices, but by choosing a different minimal-crossing projection of 8_{19} we can see that it is an algebraic link. Both of these projections and their corresponding basic polyhedra are shown in Figure 2. To avoid this type of ambiguity, we make the definition below.

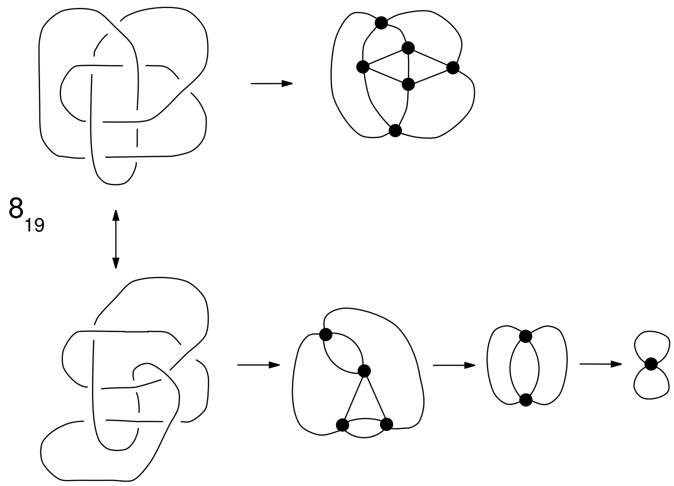

Fig. 2. Two projections of knot 8_{19}, the lower one being algebraic

Definition 1. A minimally and algebraically presentable (MAP) link is defined to be a link which possesses a minimal-crossing projection whose basic polyhedron is a figure-eight (and thus has only one vertex).

Now that we have a way to identify algebraic projections of links, we move toward finding an organizational pattern which will be efficient for piecewise-linear (PL) construction. David Gabai [13] describes how any algebraic projection can be represented by a weighted tree diagram. He defines an *arborescent link* to be "the boundary of a surface constructed by plumbing as specified by a tree." For full details on plumbing see [13], but one important point is that the class which Conway calls algebraic is the same class that Gabai calls arborescent. Next we summarize constructtion by plumbing, which results in a surface whose boundary is an arborescent link.

Given a tree and a vertex designated as the peak or center of the tree, there exists a canonical regular projection of a link associated with it up to isotopies of $\mathbb{R}^2$ [13]. An example tree is shown in Figure 3. Each weight in the tree diagram identifies an integral tangle, and its position in the tree indicates where its tangle is placed in the projection, relative to the other integral tangles. The peak indicates the primary "band," and each vertex is another band which will be attached by "plumbing." Each weight on a band indicates an integral tangle along it. In the example shown, the primary (horizontal) band has a 3 and -2 tangle along it and three vertical bands attached to it. On the left in Figure 4 is the top level of the tree (with peak indicated). Since the peak has five objects associated with it, two weights and three edges, the primary band will have five objects along it, two integral tangles and three squares where the second-level bands will be attached, as shown on the right in Figure 4. The second level of the tree is added in Figure 5, along with its plumbing diagram. Notice that the leftmost vertical band has a -1 tangle along it, and a square for the horizontal band with a 2 tangle which will be attached in the final link diagram. The completed link projection is shown in Figure 6.

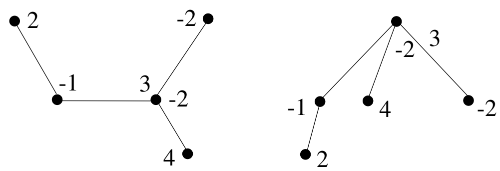

Fig. 3. A sample tree and that same tree but with a peak indicated

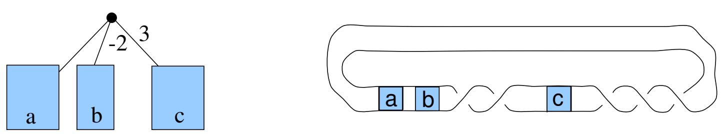

Fig. 4. Plumbing construction for the top level of the example tree

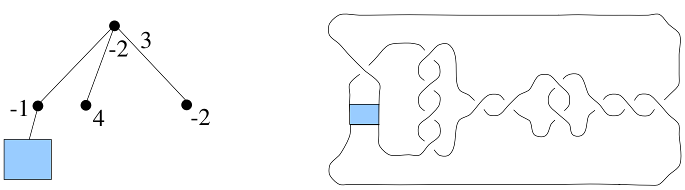

Fig. 5. Plumbing construction for the top two levels of the example tree

We will also be using two parts of a tree diagram which Gabai [13] defines as stumps and twigs: "A stump is a vertex which adjoins exactly one edge, and a twig is a vertex which adjoins exactly two edges." The simplest tree contains only stumps and twigs, as in Figure 7. Such trees represent a class of links that can be considered separately, the 2-bridge links, also called the rational links.

Edge numbers for links in the class of 2-bridge or rational links were explored in a previous paper (McCabe [5]). There it was proven that any rational knot or link K, except for the Hopf link, can be constructed with $c(K) + 3$ edges while simultaneously producing a projection of K which displays $c(K)$ crossings. Thus by Gabai's introductory material and previous results, we have that the edge number of a Hopf link is six and that $e(K) \leq c(K) + 3$ for any other arborescent link K which possesses a tree in which no vertex adjoins more than two edges. (This is just

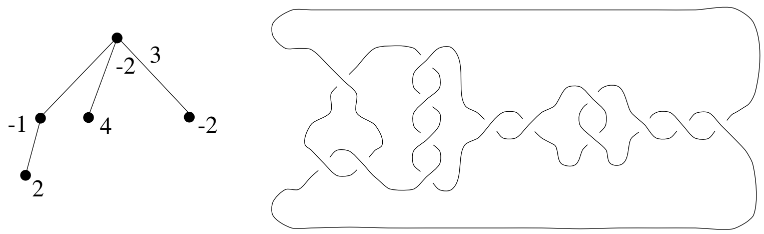

Fig. 6. Complete link projection corresponding to the example tree

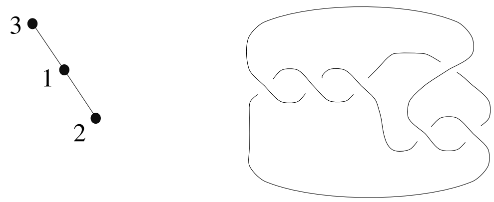

Fig. 7. A 2-bridge/rational example

another way to describe the rational links.) From here on we focus our attention on the non-rational MAP links, so we only consider links L which have the property that each tree corresponding to L has at least one vertex which adjoins three or more edges.

We are nearly ready to define a convenient and uniform layout for the trees associated with algebraic links, but first we introduce a term from Conway [12] called *flyping*. Flyping is an isotopy move on a link or tangle diagram, used by Conway and before that by P. G. Tait to flip over a portion of a link. In so doing, a single crossing moves from one side of the flipped portion to the other side. This gives a new projection for the original link. This type of change in a projection is sometimes needed in order to put tree diagrams into a consistent form. An example of flyping is shown in Figure 8.

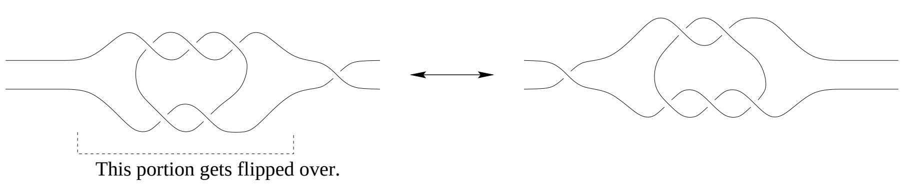

Fig. 8. Flyping

3. A Standard Form for Algebraic Projections and their Trees

What follows is new work which uses the terminology and descriptions from Section 2. We begin by defining a standard form for the tree diagrams of algebraic link projections and then showing that every link under consideration has a tree in that form. Note that while proving a link has a tree in this form, we are also showing how to create such a tree from a given link projection.

Definition 2. A tree T is defined to be in PL standard form if the following all hold:

(i) The only non-zero weights are at stumps.

(ii) Each twig in the tree adjoins a stump with weight greater than one in absolute value.

(iii) Except for possibly one zero-weight stump, there is a 1-1 correspondence between the stumps of T and the integral tangles in the canonical regular projection of its corresponding link.

(iv) There is a peak vertex in T which adjoins three or more edges and from which all edges lead downward.

(v) Each vertex adjoins no more than one of the following: stumps with weight 1, stumps with weight -1, or twigs with any weight stump. Furthermore, the projection corresponding to T has been simplified by flyping so that any twig or ±1 stump in T has been moved as far to the left as possible.

Before getting into the details of putting trees into PL standard form, we view an example of the finished product being described. To modify our main example tree from Figure 6 and put it into PL standard form takes a few steps. To satisfy part (i), the -1 weight and the top -2 and 3 weights move down to stumps adjoining twigs. For part (ii), the -1 weight then moves up to become a $+1$ on a stump next to the 2. Parts (iii) and (iv) are already satisfied by this tree since we have chosen a peak and have not broken any longer integral tangles into shorter pieces. Finally, after flyping, the -2 and 3 that were along the primary band combine to form a $+1$ tangle along the band. This could be a $+1$ on a stump adjoining a twig or a -1 on a stump directly below the peak vertex (the latter being the one to choose by part (ii)). Figure 9 shows the tree from Figure 6 with the stages of modification just described, and the version of this tree in PL standard form along with its link projection is shown in Figure 10. The path from the link projection in Figure 6 to the one in Figure 10 may be seen through flyping. The vertical -2 tangle gets flyped twice to combine the horizontal -2 and 3 tangles into a 1 tangle along the primary band. Then that 1 tangle is flyped from the right side of the projection to the left side.

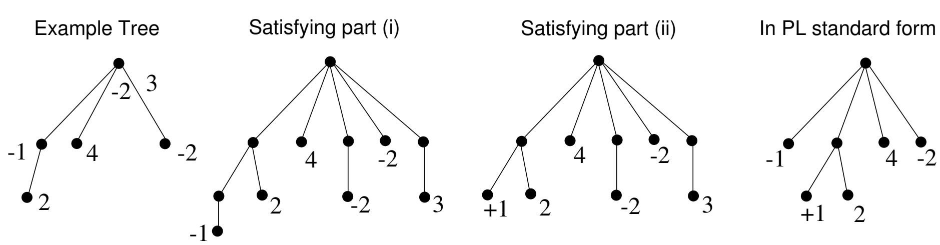

Fig. 9. Putting a tree into PL standard form

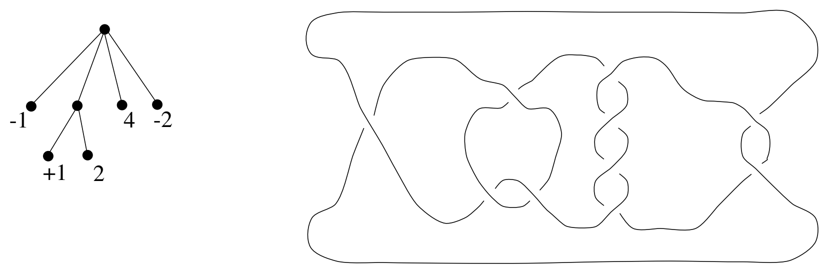

Fig. 10. Link projection corresponding to the PL standard form of the example tree

Theorem 1. For each non-rational MAP link L, there exists a corresponding tree T in PL standard form.

Proof. Let T_0 be any tree corresponding to the given MAP link L.

(i) For every non-zero weight n which is not at a stump, insert a twig which adjoins a weight n stump, as is shown in Figure 11. This changes the tree T_0 into a new tree T_1, but the projection of L corresponding to T_0 only changes by an isotopy of $\mathbb{R}^2$ when becoming the projection corresponding to T_1.

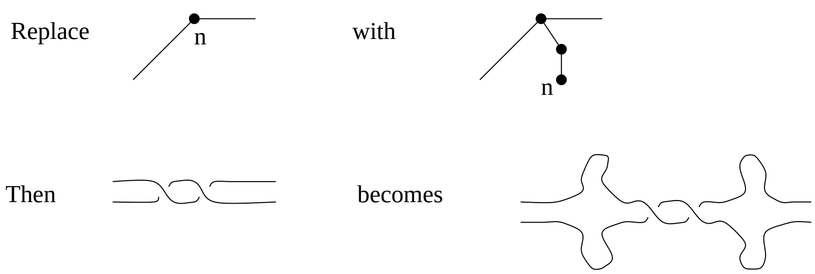

Fig. 11. Move nonzero weights to stumps

(ii) In the previous step we added twigs which will be useful, but other twigs will be viewed as unnecessary. To eliminate twigs which do not adjoin stumps, make the replacements of Figure 12 to create tree T_2. The dotted circles and triangles in that figure first represent two portions of the tree, and then represent the corresponding tangles in the link diagram.

A second type of unnecessary twig is one which adjoins a ± 1 stump. Replace each of these twig-stump combinations with a stump of weight ∓ 1 as in Figure 13. Intuitively, this works because a single crossing may look positive when viewed as being along a horizontal band, but that same crossing looks negative when viewed as being along a vertical band. After making these replacements, we have tree T_3.

A third type of unnecessary twig is one that adjoins a zero-weight stump. This represents a tangle with zero crossings along the band, so it may be removed without changing the corresponding projection. Remove all such twigs and call the resulting tree T_4.

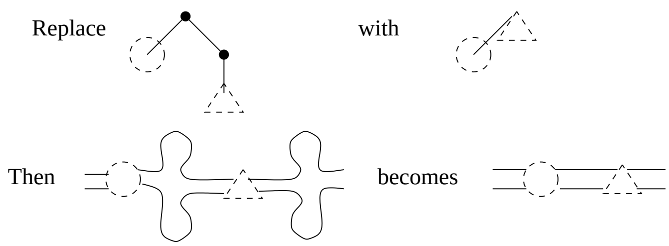

Fig. 12. Removing one type of unnecessary twig

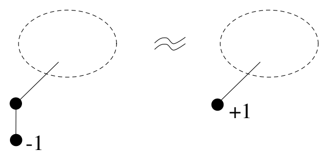

Fig. 13. Removing a second type of unnecessary twig

(*iii*) This step eliminates unnecessary stumps. If a single integral tangle is described by more than one stump, then the tree is needlessly complex. See Figure 14 for an example. Notice that the two tangle projections shown in this figure are identical up to isotopies of $\mathbb{R}^2$. We now have a tree representing L which we call T_5.

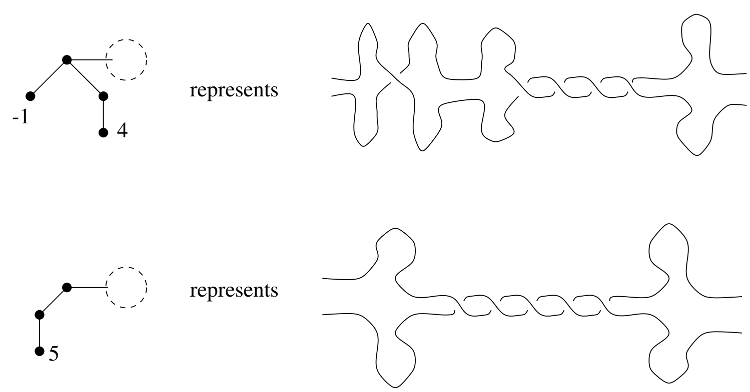

Fig. 14. Removing unnecessary stumps

Also, the entire algebraic *tangle* may be represented by a tree with no zero weight stumps, but there are two ways to close off a tangle into a link. (These two ways are sometimes called the numerator and denominator.) The tree/plumbing notation always calls for connecting across the top and across the bottom, so in order to have both options one zero-weight stump may be necessary. A row of linked circles as in a chain is an example of a link which would need a zero-weight stump. The middle row of Figure 15 shows two projections of a link requiring a zero weight, and the top row compares it to the link projection corresponding to the same tree but with the zero weight removed. These are two different link types. If there are any zero-weight stumps which are not of the sort just described, remove them to create tree T_6.

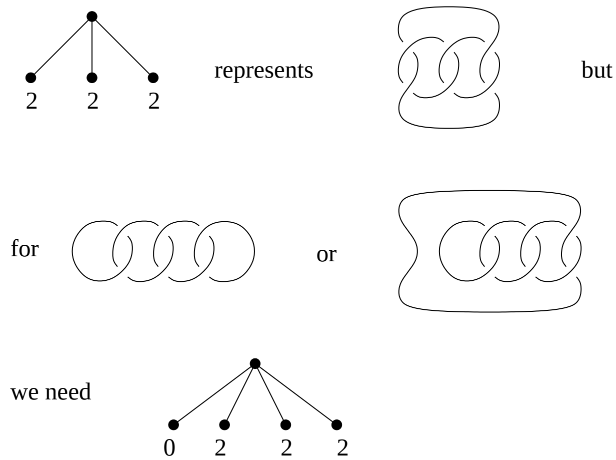

Fig. 15. Example with a necessary 0-weight stump

(iv) Since we are only considering non-rational arborescent links, there is a vertex v_0 in tree T_6 which adjoins three or more edges. By an isotopy of $\mathbb{R}^2$ (in which the tree is embedded), push this vertex in one direction (call that upward) and let all edges radiate "downward" from it. Choosing vertex v_0 to be the peak of the tree gives a reference for what we can call up, down, right, and left in tree T_6. It also indicates which is the primary band in Gabai's description of plumbing [13].

(v) If any vertex adjoins both a stump with weight ± 1 and a twig (which may adjoin any weight stump), or if it adjoins more than one of either of these, then the tree and projection can be simplified by flyping. We will flype portions of the projection so that the corresponding stumps and twigs in the tree move as far to the left as possible.

This change in the projection simplifies the tree. It can eliminate one or more stumps if more than one weight ± 1 stump adjoins the same vertex. It can combine two twigs which adjoin the same vertex (or eliminate both if they cancel out), or it can simply move a weight ± 1 stump or a twig to the left in the tree for uniformity. See Figure 16 for three examples of the effects of flyping. Once this has been done throughout the tree, we have tree T_7 which still represents a link of type L. This concludes part five of the proof.

Note that as we traveled from trees T_i to T_{i+1} in this proof, we did not invalidate any of the previous conditions. For example, tree T_5 from step three still satisfies conditions one and two from Definition 2. Thus if we let $T = T_7$, we have a tree in PL standard form which represents the given link L. □

Next we define features of trees which will ultimately be used to describe an upper bound on edge numbers of non-rational MAP links.

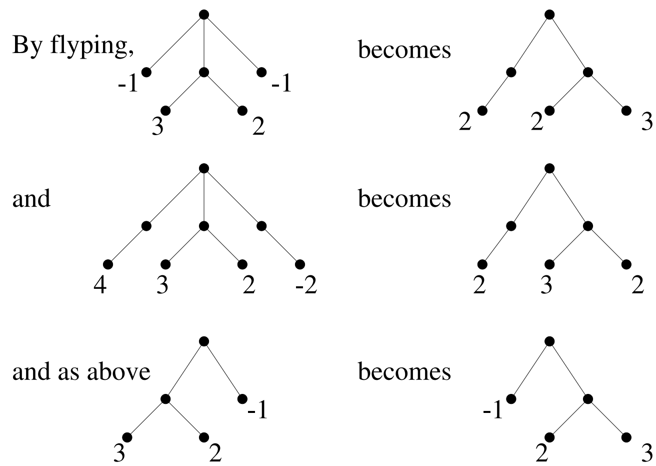

Fig. 16. Effects of flyping on trees

Definition 3. Define c_p to be the number of crossings in a projection, and let n_2 be the number of stumps in a tree which have weight 2 or -2.

The importance of the number of 2 or -2 tangles will become clear once we begin to make PL constructions. The values c_p and n_2 are defined for tangles and their trees as well as for link projections and their trees. For example, $c_p = 4$ and $n_2 = 0$ for a -4 integral tangle, and $c_p = 7$ and $n_2 = 2$ for the rational tangle or knot corresponding to the continued fraction $[3, 2, 2]$. Because of the one-to-one relationship between the stumps in a tree and the integral tangles in a projection, n_2 is equal to the number of ± 2 tangles in a projection and c_p is equal to the sum of the absolute values of the weights in a tree. The next definition introduces what will be the primary subgraphs in a tree of an arborescent link.

Definition 4. If a vertex in a tree in PL standard form adjoins only stumps and twigs with the possible exception of one edge that extends upward, and if there are at least two stumps and/or twigs extending downward from that vertex, then that vertex together with the adjoining stumps and twigs below it is defined to be a fan.

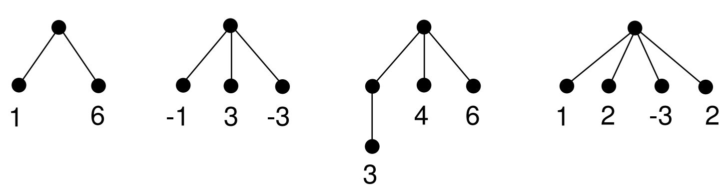

Fig. 17. Examples of fans

Notice that since simplification by flyping includes moving all weight ± 1 stumps and all twigs in a fan to the left, the leftmost stump of a fan in PL standard form will be the only one which may have weight ± 1 or which may be on a twig.

Trees are primarily made up of fans, but they may also contain stumps which are not in a fan. For two examples of trees with such stumps, see Figure 18. To describe a new upper bound on edge number, we will want to count the number of fans in a tree and the number of 1 or -1 stumps which are not in a fan, so we make the following definition.

Definition 5. Let n_f be the number of fans in a tree, and let $n_{1\notin f}$ be the number of ± 1 stumps in a tree which are not in a fan.

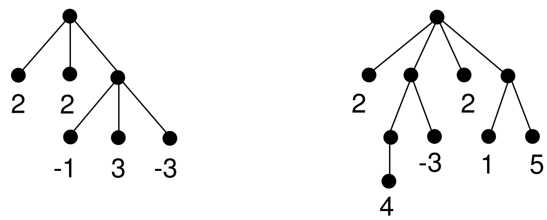

Fig. 18. Four stumps with weight 2 which are not in fans

This concludes our basic description of algebraic links and useful ways to represent them. We have seen that to achieve a bound for edge number, we consider a MAP link with a given projection. Identifying its integral tangles and how they are connected gives us a corresponding tree (as Gabai described). We then put the tree in PL standard form, identify all fans, and make the corresponding changes in the link projection. In the next section, we describe an efficient PL construction for the link.

4. Piecewise-Linear Construction

We begin with the PL construction of integral tangles, then move on to the tangles corresponding to fans, and finally to the rest of the link. As was noted in McCabe [5], any integral tangle may be constructed within a shallow rectangular box, with all four exiting edges leaving at angles between 40 and 50 degrees with the horizontal axis, as you look down on the box. The Integral Tangle Chart, Figure 19, shows three similar patterns for constructing integral tangles. Each pattern corresponds to the different positions a tangle can have in a tree. Roughly, Row 1 is used for the leftmost tangle in a fan, Row 2 is used when attaching a vertical integral tangle alongside an existing tangle, and Row 3 when attaching a vertical integral tangle below an existing tangle. Crossings are indicated in the chart by double points for simplicity, so each diagram could represent a positive or a negative integral tangle. Note that the pattern of each row extends to any number of crossings by adding extra pairs of crossings to the bottom of the 5 or 6 tangle from that row (building on the 5 for odd integers and on the 6 for even integers). Just extend the longer edge from the bottom of the appropriate tangle and add two new edges for every pair of crossings, the first horizontal and the second at a 45 degree angle with the horizontal. We use Rows 1 and 2 for the PL construction of a fan.

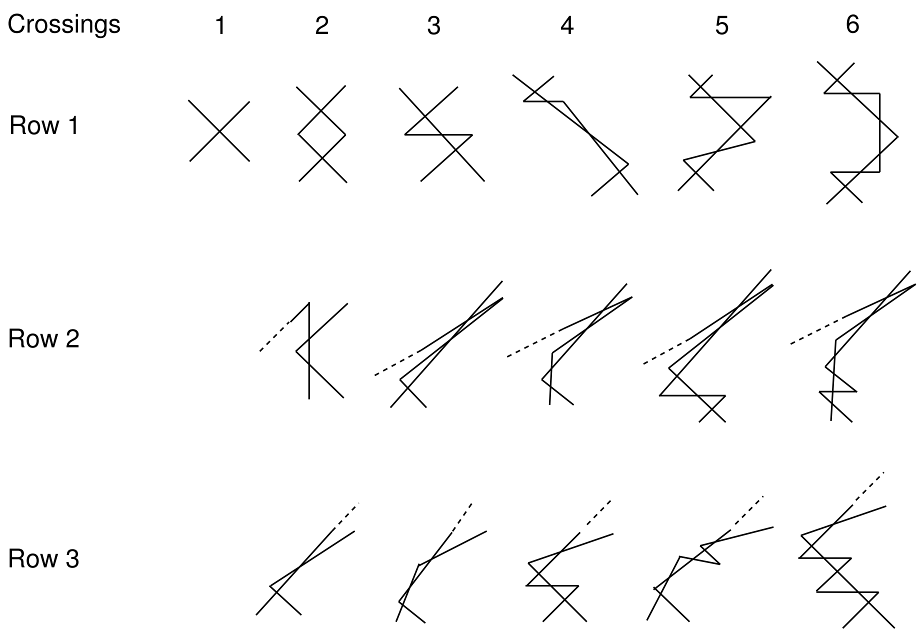

Fig. 19. Integral Tangle Chart

To construct the portion of an algebraic link which corresponds to a fan in its tree, first construct the integral tangle for the leftmost stump in the fan within a shallow rectangular box, following the diagrams in Row 1 of the Integral Tangle Chart. If this stump is on a twig, rotate its tangle 90 degrees, so it runs along a horizontal band instead of a vertical one. For any tangles from Row 1 with c_p crossings, $c_p + 1 + n_2$ edges are used. In other words, $c_p + 1$ edges are used for each integral tangle except for the ± 2 tangles, for which an extra edge is necessary to keep the exiting edges at the desired angles. This extra edge is the reason we use n_2 to count the number of ± 2 tangles.

Once the first (leftmost) tangle of a fan has been constructed, extend its upper right edge and use it as one of the edges of the next integral tangle. The second tangle is identified as the second stump from the left in the fan and is to be constructed using Row 2 of the Integral Tangle Chart. Tangles in this row require $c_p + n_2$ edges since one edge, the partially dashed edge in the figure, always comes from extending an edge from an earlier tangle. In the new algebraic tangle, which consists of two integral tangles, a total of $c_p + 1 + n_2$ edges have been used. If there is a third stump in the fan, it should be connected to the second tangle by extending the lower right edge of the second tangle and using it as an edge of the third integral tangle. Construct the third tangle as shown in Row 2 of the chart, but reflected through a horizontal line, so the dashed edge is on the lower left. Continue adding integral tangles in this manner, using Row 2 and alternating between top and bottom extending edges, until the tangle corresponding to the entire fan is completed. An example is shown in Figure 20, and the edge number result from our construction of a fan is summarized below.

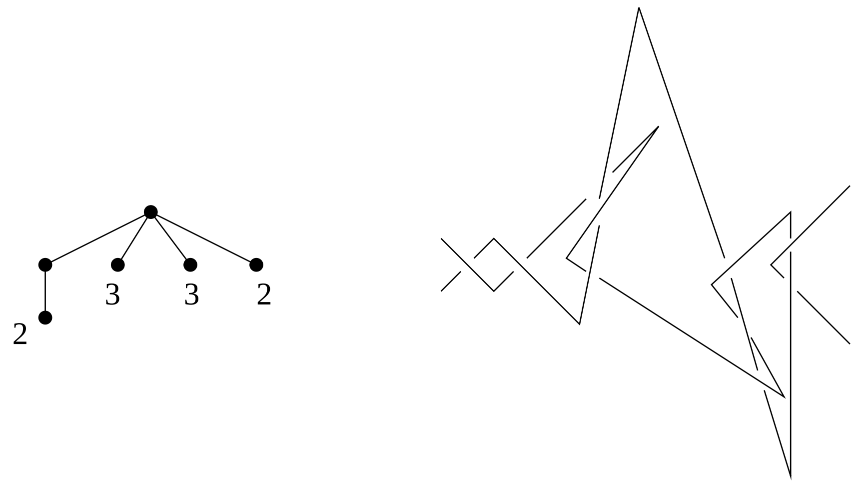

Fig. 20. A fan with its constructed tangle

Lemma 1. The tangle corresponding to a fan can be constructed with c_p+1+n_2 edges and have the four exiting edges leave at 40 to 50 degree angles with the horizontal.

In this way we construct a PL version of the tangle corresponding to each fan in a given tree. Below, a new bound on edge number for MAP links is stated, and within that theorem's proof is the rest of the PL construction for these links.

Theorem 2. For any non-rational MAP link K, edge number is bounded above by $e(K) \leq c(K) + n_f + n_2 + n_{1\notin f} + 2$.

Proof. Let K be a non-rational MAP link. By Theorem 1, there exists a tree in PL standard form which represents a minimal-crossing projection of K.

Notice that this tree must have at least one fan. Every stump or stump on a twig is "next to" something, meaning that the vertex above the stump or twig adjoins more than that one stump or twig. If that vertex is not a part of a fan, then it adjoins a vertex which is not a stump nor a twig and thus leads farther down. Continuing down the tree along such edges eventually leads to a stump or twig which has only stumps and/or a twig next to it, in other words, a fan.

We first construct the tangles corresponding to the fans in the tree, as described above. By Lemma 1, each of these tangles uses $c_p + 1 + n_2$ edges.

The entire tree may simply be one fan. Otherwise, each fan in the tree has an edge extending up from its top vertex. The vertex at the top of this edge cannot be a twig that extends upward (see part 2 of Definition 2), so it must have at least one other edge extending down from it. We say the vertex at the bottom of such an edge is next to or adjacent to the fan. It is either a stump, a twig, another fan, or a more complex portion of the tree which includes at least one fan farther down.

If a stump is next to a fan, its tangle may be attached to the fan's tangle by referring to Row 2 of the Integral Tangle Chart and extending one edge of the fan's tangle into the new integral tangle. This cannot be done if the tangle is a ± 1, however. A ± 1 tangle requires two new edges, so the new tangle's exiting edges can stay at angles between 40 and 50 degrees with the horizontal. This is why we defined $n_{1 \notin f}$ to be the number of weight ± 1 stumps which are not in a fan. Thus the number of edges in a tangle which corresponds to a fan with an adjacent stump is $c_p + 1 + n_2 + n_{1 \notin f}$. Examples of these tangles are in Figure 21. Their positions in the tree determine whether they appear as shown (two horizontal bands plumbed onto a vertical band) or rotated 90 degrees (which would be two vertical bands on a horizontal band).

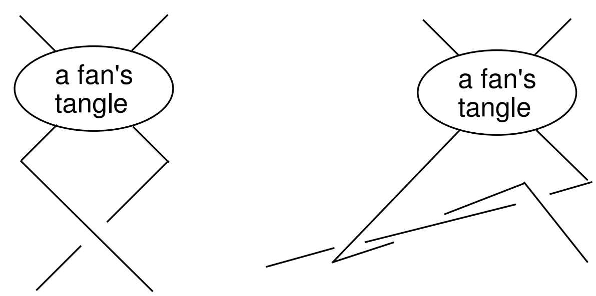

Fig. 21. Two tangles corresponding to a stump next to a fan

If a twig is next to a fan, then the tangle of its stump may be attached to the fan's tangle by referring to Row 3 of the Integral Tangle Chart and extending the lower left edge of the fan's tangle into the new integral tangle. See Figure 22. Since the PL standard form contains no ± 1 stumps on twigs, we need not consider them separately, and the value of $n_{1 \notin f}$ is zero for this portion. Constructing a fan with an adjacent twig requires no more than $c_p + 1 + n_2$ edges. (You may notice that the 2 tangle in Row 3 only requires two new edges, unlike the 2 tangles in Rows 1 and 2. Thus we could define n_2 to be the number of 2's in a tree which are not on a twig next to a fan, but this seems overly complicated.)

If two fans' tangles or more complicated tangles are next to each other, they can be connected without adding any new edges, as is shown in Figure 23. The new tangle has $c_p + n_f + n_2 + n_{1 \notin f}$ edges, and its four exiting edges are at 40 to 50 degree angles with the horizontal.

We can now describe the PL construction of the entire link K. We begin with the "lowest" fans, the ones farthest away from the peak vertex. We can construct these and then connect to each of them all stumps and twigs which are next to them. This gives new tangles, each with $c_p + 1 + n_2 + n_{1 \notin f} = c_p + n_f + n_2 + n_{1 \notin f}$ edges in all and whose four exiting edges lie at 40 to 50 degree angles with the horizontal.

One of these may represent the whole link; otherwise the vertex at the top of

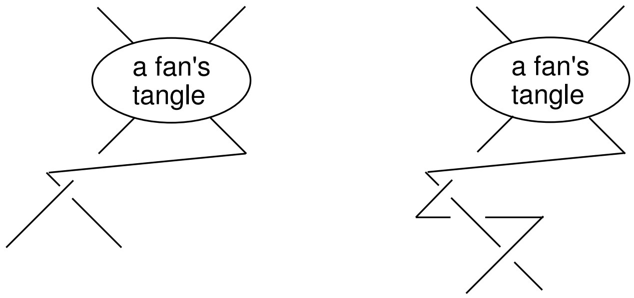

Fig. 22. Two tangles corresponding to a twig next to a fan

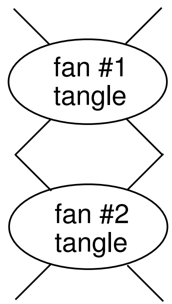

Fig. 23. Connecting the tangles of two adjacent fans

each of these portions of the tree will have an edge extending up from it. Next to each such edge will be at least one edge which leads to a stump, twig, fan, or one of the tangles already constructed. Connect these as described above. This gives new tangles with $c_p + n_f + n_2 + n_{1\notin f}$ edges and the same type of exiting edges. Continue connecting in this manner until the entire tangle is completed. There may be a weight zero stump just off of the peak, which is equivalent to an instruction to connect up the final four exiting edges in the way which does not naturally follow from the plumbing. Either way, two new edges close the tangle up into a link, so $e(K) \leq c(K) + n_f + n_2 + n_{1\notin f} + 2$ as desired. □

To demonstrate an example of the construction of non-rational MAP links, we work with $K = 10_{48}$, which has a minimal-crossing projection that is algebraic. The projection of K shown in Figure 24 corresponds to the tree next to it, which is in PL standard form. The fan (with a -1 and a -4 tangle) would be constructed first. Then the 3 tangle would be attached as shown in Row 2 of the Integral Tangle Chart and in Figure 21, and then a 2 tangle would be attached in the same way. Finally, two edges would be added to close up the knot. The resulting PL knot is shown in Figure 25. It has $c(K) + n_f + n_2 + n_{1\notin f} + 2 = 10 + 1 + 1 + 0 + 2 = 14$ edges, so $e(10_{48}) \leq 14$.

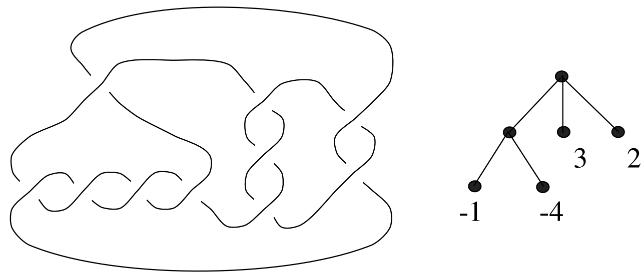

Fig. 24. The MAP knot 10_{48} and its tree

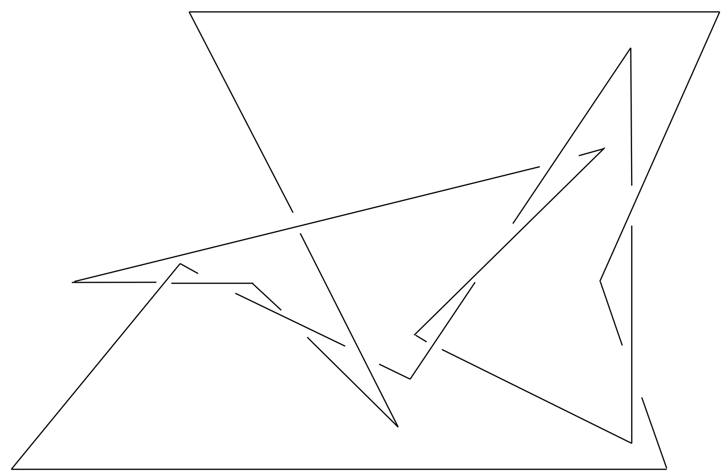

Fig. 25. The PL construction for the knot 10_{48}

5. Comparisons to Known Bounds

We would like to compare the upper bound of Theorem 2 to Negami's bound of $e(K) \leq 2c(K)$ [1]. Before going further, let us rearrange the expression and define new terms.

Definition 6. Let n_1 be the number of weight ± 1 stumps in a tree, and let $n_{f \not\ni 1}$ be the number of fans which do not have any stumps of weight ± 1.

Notice that $n_f + n_{1 \notin f} = n_1 + n_{f \not\ni 1}$, so by Theorem 2 we have $e(K) \leq c(K) + n_1 + n_2 + n_{f \not\ni 1} + 2$ for any non-rational MAP link K.

At least one such link has edge number equal to twice its crossing number, so we cannot expect to get $e(K) < 2c(K)$ in general. One example is the four-component chain link from Figure 15. It has crossing number six and requires $12 = 2c(K)$ edges, three for each component. This edge number is precisely the number given by our new bound since $c(K) + n_1 + n_2 + n_{f \not\ni 1} + 2 = 6 + 0 + 3 + 1 + 2 = 12$. Next we show that the new bound is less than or equal to $2c(K)$ for any non-rational algebraic link K which has an algebraic minimal-crossing projection.

Lemma 2. Let K be a non-rational MAP link, and let its tree T be in PL standard form. Then the following inequalities hold, where $c = c(K)$:

$$(i) \quad n_2 \leq \frac{1}{2}(c - n_1)$$

$$(ii) \quad n_{f\not\ni 1} \leq \frac{1}{4}(c - n_1)$$

$$(iii) \quad n_1 \leq c - 6$$

Proof. (i) Define n_i to be the number of $\pm i$ weights in the tree T, and let k be the largest absolute value of all the weights in T. Since the crossing number equals $n_1 + 2n_2 + 3n_3 + \cdots + kn_k$, we have $c \geq n_1 + 2n_2$, so the desired result follows.

(ii) Consider a fan with two stumps, each with weight 2 or -2. This is the fan whose tangle has the least number of crossings possible while having no ± 1 tangles. Thus each tangle corresponding to a fan in T with no ± 1 stumps has at least four crossings. For the entire tree T we have $4n_{f\not\ni 1} + n_1 \leq c$, which gives the desired result immediately.

(iii) We show that there must be at least three stumps with weights of absolute value greater than one. Recall that it was noted previously that any twig which was connected to a stump with weight ± 1 is unnecessary since a ± 1 on the end of a twig is equivalent to a ∓ 1 on the end of a stump in the same position. Similarly, we could take our tree out of PL standard form using the equivalence shown in Figure 26.

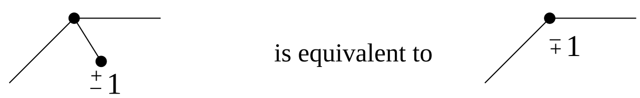

Fig. 26. Weight ± 1 stump equivalences

So if the tree T has 1 or -1 weights on all but two of its stumps, it is equivalent to a tree in which no vertex adjoins more than two edges. Only the two stumps which potentially do not have weights ± 1 will remain stumps. This would imply that K is a rational link, contrary to our assumption. Thus T must have at least three stumps which do not have weights ± 1.

If one of the three were a zero weight stump, we would have a rational tangle closed-up into a link in the non-standard direction. But even this simplifies to a smaller rational tangle since one of its integral tangles would unravel. Thus there must be at least three stumps in T which have weights greater than or equal to 2 in absolute value. Therefore we have $n_1 + 2 + 2 + 2 \leq c$, and $n_1 \leq c - 6$. $\square$

Theorem 3. For any non-rational MAP link K, the upper bound on $e(K)$ from Theorem 2 is less than or equal to $2c(K)$.

Proof. By the edge number result from Theorem 2, the equality following Definition 6, and the inequalities from Lemma 2, we have:

$$
\begin{aligned}
e(K) &\leq c + n_f + n_2 + n_{1\notin f} + 2 \\
&= c + n_1 + n_2 + n_{f\not\ni 1} + 2 \\
&\leq c + n_1 + \frac{3}{4}(c - n_1) + 2 \\
&= \frac{7}{4}c + \frac{1}{4}n_1 + 2 \\
&\leq \frac{7}{4}c + \frac{1}{4}(c - 6) + 2 \\
&= 2c + \frac{1}{2}
\end{aligned}
$$

Since $e(K)$ and $c = c(K)$ are integers, we have $e(K) \leq 2c(K)$. □

This new bound can be calculated fairly easily for links whose Conway notation is known. A tree can be made from the Conway notation, and then the values for n_f, n_2, and $n_{1\notin f}$ can be read from the version of the tree in PL standard form and inserted into the expression from Theorem 2. For non-rational MAP knots through 10 crossings, some of the better bounds are $e(K) \leq c(K)+3$ for knots 9_{35}, 9_{46}, 10_{61}, 10_{64}, 10_{139}, 10_{140}, and 10_{143} and $e(K) \leq c(K) + 4$ for knots 8_5, 8_{19}, 9_{16}, 10_{46}, 10_{48}, 10_{62}, 10_{124}, 10_{126}, 10_{141}, and 10_{142}. None of these are known to be minimal and most are not minimal (see Rawdon and Scharein [6]), but this new upper bound expression improves with links having fewer 1 tangles, 2 tangles, and fans, which is more often seen with higher crossing number. Pretzel links, for example, have only one fan. Theorem 2 shows that a pretzel link with no 2 tangles has $e(K) \leq c(K)+3$.

For an example with higher crossing number, consider the non-rational MAP knot in Figure 27. Its crossing number is 24 (since it is alternating), and the projection shown corresponds to the tree below it, which is in PL standard form. From the tree we see that $e(K) \leq c + n_f + n_2 + n_{1\notin f} + 2 = 24 + 2 + 0 + 0 + 2 = 28$.

6. Summary of Algebraic Cases

We now have results for algebraic links and tangles. For algebraic links which are not rational, $e(K) \leq 2c(K)$ was proved, so this new procedure gives bounds at least as good as the general result stated by Negami in [1]. Also, a longer expression was given in Theorem 2 which gives a better bound on edge number in many cases. For algebraic links (other than the Hopf link) which also happen to be rational, we have the result from McCabe [5] that $e(K) \leq c(K) + 3$.

For algebraic tangles we again consider rational and non-rational cases separately. Rational tangles can be constructed with $c_p + 2$ edges, and algebraic non-rational tangles can be constructed with $2c_p - 2$ edges. The latter result follows directly from the link result of Theorem 3 since two edges were used to close a tangle up into a link.

The tangle results are a step toward a new upper bound on edge number for all links since algebraic tangles are found in all link projections. What remains to be

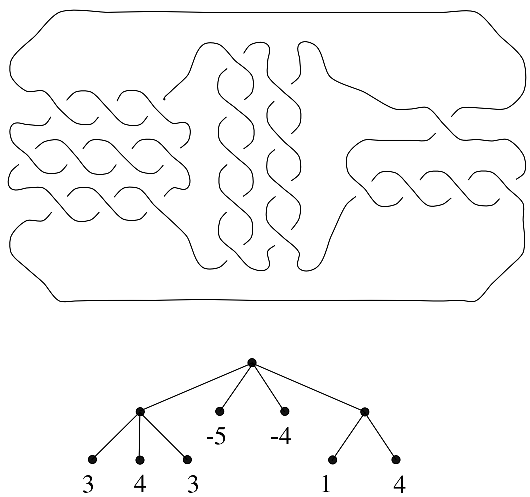

Fig. 27. A 24-crossing alternating knot and its tree

found is a PL pattern for basic polyhedra (Conway [12]) or a bound on their edge number when their 4-valent vertices have nearly perpendicular edges emanating from them. Early indications are that this new general bound could be strictly less than twice the crossing number.

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