AN UPPER BOUND ON EDGE NUMBERS OF 2-BRIDGE KNOTS AND LINKS

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ABSTRACT

A method is given for economically constructing any 2-bridge knot or link K. This construction gives an upper bound of $c(K)+3$ for the edge number of K, except in the case of the unlink and the Hopf link, and has the added advantage of realizing a projection with $c(K)$ crossings.

Keywords: knot, link, 2-bridge, edge number, integral tangle

1. Introduction

As the search for the minimal number of edges necessary to construct knots and links continues, the edge numbers $e(K)$ for more knots and links are found as are better upper and lower bounds on edge number for various classes of knots and links. The most general results are

$$\frac{5+\sqrt{25+8(c(K)-2)}}{2} \leq e(K) \leq 2c(K)$$

for any nontrivial knot or link K other than the Hopf link, where $c(K)$ is the minimal crossing number of K; see Negami [1]. Precise edge numbers for specific knots with small crossing number are known due to work by Randell [2], Meissen [3], and Millett [4]. Adams et al. [5] found precise edge numbers for compositions of trefoils and for links with small crossing number, and Jin [6] found upper bounds on edge number for torus knots and their compositions.

Here an upper bound of $c(K)+3$ will be proved for all but two of the 2-bridge knots and links. This bound is better than the general upper bound of $2c(K)$ in all cases except for the trefoil and is sharp in at least a few cases. The edge number of the trefoil, the figure-eight, and the two five-crossing knots is known to be $c(K)+3$ [2]. Although this bound is not sharp in other cases (the six and seven crossing knots, for example [3]), the method used to find it gives a projection which shows the minimal number of crossings for the knot or link. It is not apparent that this is possible in other constructions of knots and links which give bounds on edge number.

2. Standard Projections

Before describing a piecewise-linear construction for a 2-bridge knot or link K, we would like to have a standard way to visualize and describe K. Since the class of 2-bridge knots and links coincides with the class of nontrivial rational knots and links as described by Conway [7], and since every 2-bridge knot or link also has a 4-plat representation, there are many ways to describe a given K. According to Burde and Zieschang [8] (pages 181-189), there is a bijection between the classes of unoriented 2-bridge knots and links and the rational numbers between zero and one. The following definition introduces the notation we will use to represent this correspondence.

Definition 1: Let B(α,β) be the (unoriented) 2-bridge knot or link which has $\frac{\beta}{\alpha}$ as its associated rational number, where α and β are integers with the properties $0 < \beta < \alpha$ and $gcd(\alpha, \beta) = 1$. Furthermore, since each rational number can be expanded into a continued fraction, let this be represented by $\frac{\beta}{\alpha} = [a_1, a_2, \ldots, a_m]$ where the integers a_i are the quotients of the continued fraction.

Note that the conditions on α and β make them unique, but the continued fraction expansion is not. Also, assuming that α is greater than one, as we have above, excludes the unknot and the unlink from consideration.

Braid notation is also useful, in combination with 4-plats. Let the braid generators be $\{\sigma_1, \sigma_2, \sigma_3\}$ with σ_i denoting one positive (right-handed) crossing between strands i and $i+1$. We will always make the strands vertical and orient them downward to determine the signs of the crossings between strands. Once the knot or link is completed, the conventional sign of any particular crossing may be different from what it is in the braid. When we want to represent B(α, β) as a 4-plat with a defining braid $\sigma_2^{a_1}\sigma_1^{a_2}\cdots\sigma_2^{a_m}$, we put a_1 crossings between strands 2 and 3, then a_2 crossings between strands 1 and 2 below the previous crossings, etc. Since there are no σ_3's in the braid expression, there are simply no crossings between strands 3 and 4. In a 4-plat diagram, this leaves the fourth strand vertical. For example, the 4-plat with defining braid $\sigma_2^{-2}\sigma_1^{1}\sigma_2^{-3}$ is shown in Figure 1.

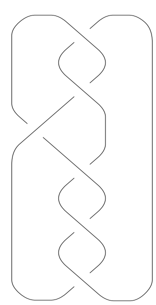


Fig. 1. A 4-plat diagram defined by $\sigma_2^{-2}\sigma_1^{1}\sigma_2^{-3}$

Now we are ready to describe the standard projection for a 2-bridge knot or link B(α, β) which will be useful for the piecewise-linear construction. The following is a property of 2-bridge knots and links given by Burde and Zieschang [8] (page 189), the notation of which has been slightly altered for consistency.

Property 1: The knot or link $B(\alpha, \beta)$ as defined above has a presentation as a 4-plat with a defining braid $\sigma_2^{-a_1}\sigma_1^{a_2}\cdots\sigma_2^{-a_m}$, with $a_i > 0$ and m odd, where the a_i are the quotients of the continued fraction $[a_1, a_2, \ldots, a_m] = \frac{\beta}{\alpha}$.

Since the continued fraction corresponding to a rational number is not unique, the first step in arriving at Property 1 is to make all the integers a_i positive. Then m may be even or odd, so use the fact that $a_m = a_m - 1 + \frac{1}{1}$ to change $[a_1, a_2, \ldots, a_m]$ into $[a_1, a_2, \ldots, a_m - 1, 1]$ whenever m is even. Figure 2 shows three typical examples of $B(\alpha, \beta)$.

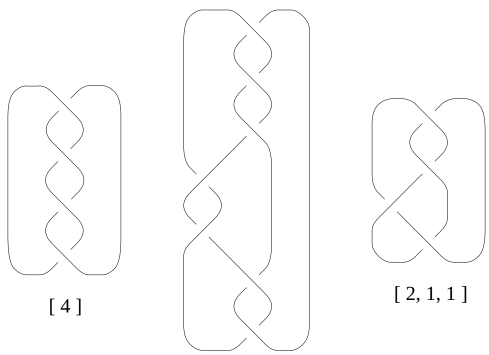


Fig. 2. Standard projections

Note that this gives an alternating presentation of $B(\alpha, \beta)$. This, along with a theorem proved by Kauffman [9], Murasugi [10], and Thistlethwaite [11] in 1987, shows that the diagram from Property 1 will always display the minimal number of crossings. The theorem may be stated as: Any simple, alternating projection of a given link K displays the minimal number of crossings, $c(K)$. If we look at the diagram from Property 1 as a projection which does not distinguish between over- and under-crossings but only shows double points, then we can see that such a diagram will be connected and that at each crossing, no two of the four local regions will be part of the same region of the plane in the larger diagram. Satisfying these two conditions is what it means to be simple. Thus the diagrams from Property 1 are simple and alternating, so they are indeed minimal crossing projections, as desired.

3. Basic Building Blocks - Integral Tangles

Each σ_i^k in the braid expression from Property 1 contributes a series of half-twists to the projection, and these series of half-twists correspond to integral tangles in Conway's terminology [7]. A convenient way to construct 2-bridge links is one integral tangle at a time, so here we observe various constructions of integral tangles.

Integral tangles are named by the number of half-twists (i.e. crossings) and by the direction of those twists, where left-handed crossings are labeled negative and right-handed ones positive. There will be no ambiguity about +1 versus -1 tangles here since we will be viewing all twists as vertical. Thus, for example σ_i^3 contributes a 3 tangle to the diagram.

Now observe that any integral tangle n may be made with $|n| + 1$ edges as is shown in Figure 3.

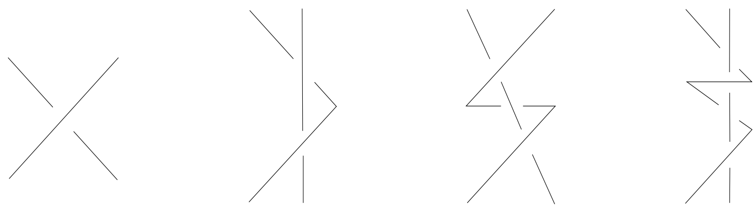


Fig. 3. Integral Tangles

The negative integral tangles are simply the mirror images of the positive ones. In this scheme, the tangles with n even have one vertical edge running from top to bottom, and those with n odd have one diagonal edge which runs from top to bottom. The remaining edges zig-zag around the first edge. Although these zig-zagging edges tend to turn around the central axis as more edges are added, we can build any integral tangle n with $|n| + 1$ edges while preserving the types of projections shown above which have $|n|$ crossings.

The turning motion can be seen by looking at consecutive endpoints of the zig-zagging edges. Three consecutive endpoints, p_1, p_2, and p_3, define a plane P_{123}, but there is no way to make p_4 lie on that plane. However, p_4 can be made as close to P_{123} as we wish. Using this idea while traveling down the tangle, we can see that any integral tangle can be constructed in 3-space within an arbitrarily shallow rectangular box where $\epsilon > 0$ is the depth of the box (which is perpendicular to the plane of the paper) and where the bottom face gives a plane for a projection of the type shown above.

Further properties of these integral tangle constructions are that all exiting edges of a tangle except for the vertical edges in even integral tangles can be made to have slopes 1 or -1 in the plane of projection. We will sometimes only make the first and last of the zig-zagging edges have slopes 1 or -1, however. The vertical edge of an even tangle or the long diagonal edge of an odd tangle may be tilted slightly to the right or left if necessary. Finally, notice that the even tangles shown above all have their vertical edges on the right side of the tangle. This edge could just as easily be on the left, as in Figure 4.

4. An Upper Bound

Now we are ready to construct the entire 2-bridge knot or link B(α, β) in the desired form. The projection mentioned in the theorem that follows will simply be a piecewise-linear version of the one described in Property 1.

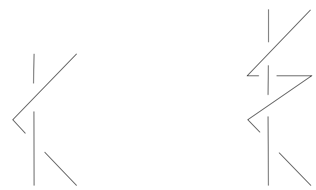


Fig. 4. More Integral Tangles

Theorem
Any 2-bridge knot or link K except for the unlink and the Hopf link can be constructed with $c(K)+3$ edges while simultaneously producing a projection of K which displays $c(K)$ crossings.

Notice that the unknot requires three edges, so the 2-component unlink and the Hopf link require three edges for each component. Thus as is shown in Figure 5, $e(unlink) = 6$ and $e(Hopf) = 6 = c(Hopf) + 4$.

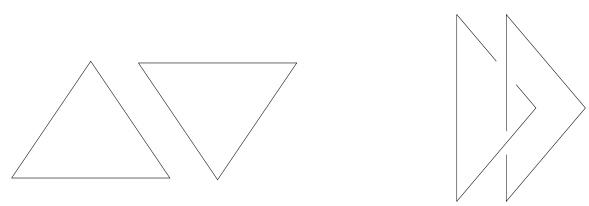


Fig. 5. The Two Exceptions, an unlink and a Hopf link

Proof of Theorem
Let K be a knot or link whose bridge number is 2. Assume K is not the unlink or a Hopf link. From Property 1, K has associated with it a positive rational number $\frac{\beta}{\alpha}$ and a continued fraction $[a_1, a_2, \ldots, a_m] = \frac{\beta}{\alpha}$ where $a_i > 0$ and m is odd. Also, K has a 4-plat representation with defining braid $\sigma_2^{-a_1}\sigma_1^{a_2}\cdots\sigma_2^{-a_m}$. Since the $m = 1$ case is different from the general case, it will be considered later.

The General Case:
Assume $m > 1$. Construct a $-a_1$ tangle as described in Section 3. If $-a_1$ is even, put its long vertical edge to the right, as shown in Figure 3. Extend the lower left edge of the $-a_1$ tangle, called edge e_{12}, and use it as the upper right edge of the a_2 tangle, which is to be built next. If a_2 is even, put its long vertical edge to the left. This gives $\sigma_2^{-a_1}\sigma_1^{a_2}$ with $a_1 + a_2 + 1$ edges. See Figure 6 for an example.

Now, we connect the upper left edges of tangles $-a_1$ and a_2. Since these integral tangles were constructed to fit within shallow rectangular boxes, extending the two edges will bring their endpoints close to one another. If both a_1 and a_2 are odd, the edge from the $-a_1$ tangle can be made more horizontal first, to prevent the two

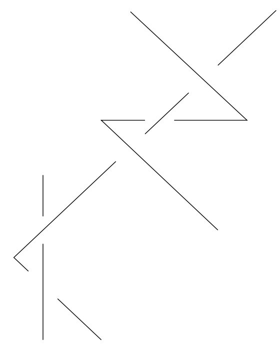


Fig. 6. First step example with $a_1 = 3$ and $a_2 = 2$

edges from being parallel. Once the two endpoints are close, they can be made to meet by slightly rotating the a_2 tangle about the axis of edge e_{12}.

A similar process is followed to attach a $-a_3$ tangle. Edge e_{23}, the lower right edge of the a_2 tangle, is extended and used as the upper left edge of the $-a_3$ tangle. If $-a_3$ is even, its long vertical edge goes to the right. Once the $-a_3$ tangle is constructed, its upper right edge is to meet the lower right edge of the $-a_1$ tangle. These two edges are extended until their endpoints are close, and then the $-a_3$ tangle is rotated slightly about the e_{23} axis until the endpoints meet. An example of what we have constructed so far is shown in Figure 7. Note that $a_1 + a_2 + a_3 + 1$ edges have been used.

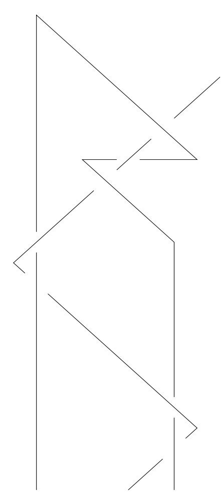


Fig. 7. Intermediate step example with $a_1 = 3$, $a_2 = 2$, and $a_3 = 2$

If $m = 3$, we are nearly done. If $m > 3$, we continue adding tangles as we have with a_2 and $-a_3$. Even subscripts correspond to positive twists and odd subscripts to negative ones, as is shown in the braid representation. Even a_i tangles always have their long vertical edges towards the outside of the construction, and these long edges may be tilted slightly off the vertical if necessary. The upper left edge of an even-subscripted tangle always meets the lower left edge of the previous even-subscripted tangle, which is above it. Similarly, the upper right edge of an odd-subscripted tangle meets the lower right edge of the previous odd-subscripted tangle.

Once the m integral tangles have been constructed and connected, all that remains is to connect the four free edges to form the knot or link. Note that $a_1 + a_2 + \cdots + a_m + 1$ edges have been used thus far. Figure 7 shows the positions of the four free edges in the case where K has $[3, 2, 2]$ as its continued fraction. For any K there will be one edge in the upper right corner and three edges along the bottom. If $-a_1$ is an even integer, its long edge should be given a large but finite positive slope. In order to get the desired 4-plat representation, this edge should be connected to the lower right edge by a new long edge which forms the fourth plat. These upper and lower right edges can be extended until the new edge can be placed between them with no interference from the rest of K. Then one edge is added to the bottom of K which connects the two remaining free edges. This gives the desired 4-plat representation of K which displays $c(K)$ crossings and uses $c(K) + 3$ edges (since $a_1 + a_2 + \cdots + a_m = c(K)$). See Figure 8 for the completed $[3, 2, 2]$ example.

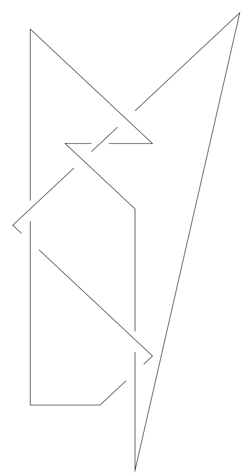
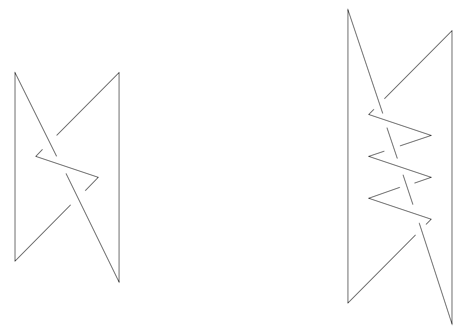


Fig. 8. The completed $[3, 2, 2]$ example

The $m = 1$ Case:

Assume $m = 1$. If a_1 is odd, construct tangle $-a_1$ as in Section 3, Figure 3. This accounts for $c(K) + 1$ edges since $c(K) = a_1$. Extend the four exiting edges until they reach past the rest of the tangle and then connect the two right endpoints with a new edge and the two left endpoints with another. This completes the knot, uses $c(K) + 3$ edges, and preserves a minimal crossing projection. See Figure 9.

If $a_1 = 2$, a Hopf link would be formed, so we disregard that case. If a_1 is an even integer greater than 2, then we need to describe a new construction for the tangle $-a_1$ which also uses $a_1 + 1$ edges. Once this is done, the tangle is closed

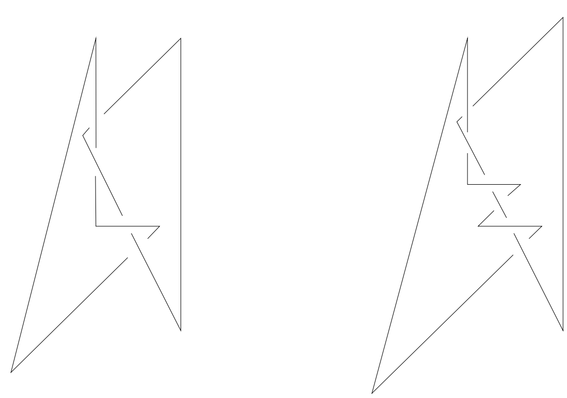


[3] [7]

a trefoil the knot 7_1

Fig. 9. Examples with $m = 1$ and a_1 odd

up into a link with two new edges just as in the a_1 odd case. The problem with the even case as it stands now is the long vertical edge. Its endpoints are the two right endpoints of the tangle, so they cannot be connected with just one new edge to close up the link. The new construction, shown in Figure 10, breaks the one long vertical edge into two edges. An even integral tangle n, with $|n| > 2$ can always be made so that its projection shows an edge with slope 1 in the lower left corner and an edge with slope -2 in the lower right corner. This lower right edge has $|n| - 2$ edges zig-zagging around it (including the lower left edge just mentioned). Then a vertical edge is attached to the top endpoint of the zig-zagging edges which will cross the long edge from the lower right. Finally, a new edge with slope 1 crosses the vertical edge to form the last crossing. This completes the tangle in such a way that its four outer edges can be extended, and then the right two and the left two can be connected by a total of two new edges to form the link $[n]$. Two examples are shown in Figure 10.



[4] [6]

Fig. 10. Even examples with $m = 1$

This construction always has $|n| - 2$ zig-zagging edges, two edges which have endpoints on the right side of the tangle, one vertical edge (in the tangle), and two final connecting edges, so $c(K) + 3$ edges are used, as desired. Also, there is a minimal crossing projection. This completes the $m = 1$ case and the proof of the theorem. □

Thus $c(K) + 3$ is an upper bound for the edge number of any nontrivial 2-bridge knot or link K, other than a Hopf link. Although Jin [6] had a similar upper bound for some 2-bridge links, and in a recent paper by Furstenberg et al. [12] some are found to have an upper bound of $c(K) + 2$, the result presented here gives standard projections with minimal crossing number and is built upon the basic integral tangles. These can be seen as the basic building blocks of any knot or link. Work on a more general result for algebraic (i.e. arborescent) links, which may in turn lead to one for all knots and links, is in preparation.

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